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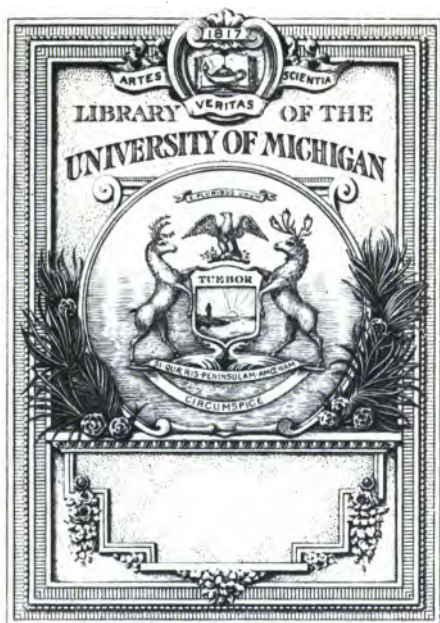
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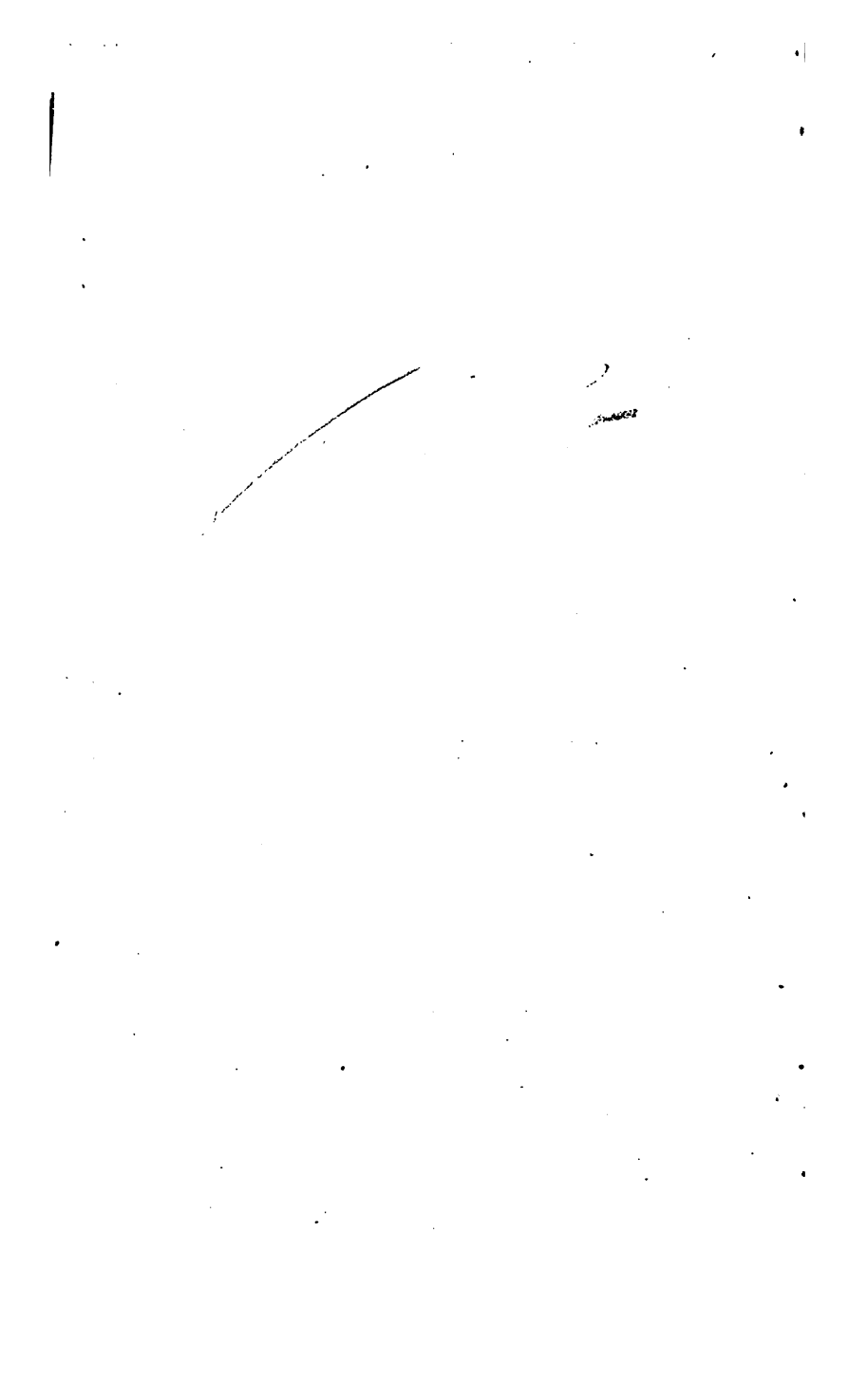
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.CURVES AND SURFACES.

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ANALYTICAL GEOMETRY,

PART SECOND,

CONTAINING

THE THEORY OF

CURVES AND SURFACES

OF THE

SECOND ORDER.

BY
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SECOND EDITION.



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1838.

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PREFACE.

② 10-31-40. H. S. I.

THIS work is to be regarded as the SECOND PART of a general Treatise on Analytical Geometry; intended, in conjunction with the preceding volume on the Conic Sections, to present a comprehensive view of the doctrine of Coordinates, as applied to the theory of Curves and Surfaces of the Second Order.

Although in the main distinct, yet a considerable portion of this second part is immediately connected with the subjects of the former treatise; and merely supplies the additions, indispensably necessary, in order to render that treatise a complete exposition of the general theory of the Conic Sections. To develop the several geometrical properties of these curves from the algebraical equations which were found to embody their characteristic peculiarities, is the sole object to which the preceding volume is devoted. The problem there discussed is:—Given the curve to investigate its analytical representation; and thence, by the operations of algebra,

to evolve its geometrical theory. The converse of this problem remains to be considered:—Given the analytical representation, to determine the curve,—in form,—in magnitude,—in position.

The importance of this latter problem, in all those applications of algebra to Geometry in which a *locus* is the object of search, must be obvious to the student; and its value in many physical enquiries of the greatest interest will appear, when he considers that the path described by any moving body is always dependent upon those mechanical laws and restrictions in obedience to which it moves; and that, when these governing conditions are expressed in the language of analysis, equations present themselves, in which the form, and extent, and position, of the orbit are all implied. To announce these, however, in ordinary language, we must, of course, be able to interpret the symbolical expressions under which the information is delivered. It is with this business of interpretation that the first two chapters of the present work are occupied; the subject of examination being the general indeterminate equation of the second degree, under every variety of form in which it can possibly present itself.

The objects proposed in these two chapters are, first, to prove that whatever be the particular equation, the only

curve represented by it must be a conic section; secondly, to investigate the analytical conditions by which each of the three curves are distinguished from the others; and lastly, when the form of the curve is thus ascertained, to determine its individual magnitude and position.

As there are two methods of arriving at this final determination, so two chapters have been devoted to the enquiry. In the first, the particular curve is discovered by means of successive transformations of coordinates, which at length bring the axes of reference into coincidence with the principal diameters, or axes, of the sought curve; whence its position and magnitude are immediately inferred. In the second chapter the formulas of transformation are dispensed with; and the individual curve determined by a direct examination, or *discussion*, as it is called, of the proposed equation. These, with two supplementary chapters,—one showing the geometrical applications of the preceding discussion, and the other containing a miscellaneous collection of important and interesting analytical truths,—occupy the first section of the book, and complete the general theory of Curves of the Second Order.

The remaining portion of the work is devoted to Geometry of Three Dimensions; or to the theory of Lines and Surfaces in Space. Of the Surfaces of the Second Order, I have

presented a very full examination. They have long been objects of interesting contemplation to the mathematician, as well on account of the beautiful properties which belong to them, as because of the intimate relationship which is found to exist between this class of surfaces and all those of a more complicated form;—the whole theory of the curvature of surfaces in general, being ultimately referrible to that of the surfaces of the second order. The study of surfaces is moreover becoming daily of increased importance, in consequence of their connexion with the recent researches into the physical laws of Light; and it is hoped that what is here offered will serve as a basis, sufficiently broad, for the support of those more advanced theories, which a higher Calculus has enabled analysts to construct.

BELFAST COLLEGE;
April, 1838.

Shortly will be published,

THE ELEMENTS OF EUCLID,

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WITH NOTES, ILLUSTRATIONS, AND CORRECTIONS,

A simplification of the fifth and sixth books, and Appendix on Plane Trigonometry.

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BY J. R. YOUNG.

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ERRATA.

Page 3, line 7, for $= \frac{P}{A^3} x^2$, read $+ \frac{P}{A^3} x^2$.

4, ... 14, for x read x^2 .

67, ... 12, for x read x^2 .

75, ... 13, for $2B^2$, read $2A^2B^2$.

91, ... 18, for E, e , read E^2, e^2 .

116, in the diagram, for g , read p .

144, line 3 from bottom, for α' , read β' .

146, ... 4 from bottom, for U' , read $\cos U'$.

ADDITIONAL ERRATA IN PART I.

Page 47, line 12, for c , read x'' .

... 92, ... 5 from bottom, for B , read B^2 .

... 116, ... 6, change the *minus* in numerator and denominator into *plus*.

ANALYTICAL GEOMETRY.

SECOND PART.

SECTION I.

(150.) THE preceding part of the present treatise has been chiefly devoted to a discussion of the form and properties of the Conic Sections; and the various results that we have obtained have been deduced from an analysis of the several equations which these curves have furnished when the law of their geometrical description upon a plane came to be expressed in the language of Algebra. The equations which have thus embodied the peculiar characteristics of each curve, though differing from one another in form, all belong to the same class—the second order; but as yet the student has had no reason to suspect that every equation of this order of whatever form, provided only it contain two variables, must be the analytical representative of one or other of the three curves. Such, however, is the case; and it is the object of the present section to establish this remarkable connexion between indeterminate equations of the second degree and the Conic Sections; and thus to prove that they comprise all the curves of the second order.

CHAPTER I.

ON THE LOCI OF INDETERMINATE EQUATIONS OF THE SECOND DEGREE.

(151.) In order to show that the curve, analytically represented by any indeterminate equation of the second degree, must be one or other of the three already discussed, we shall, in this chapter, proceed as follows. We shall first prove that any equation whose form agrees with that characterizing one of the conic sections, must have that curve for its locus; and next, that any indeterminate equation of the second degree, of whatever form, may be reduced to one or other of those particular forms previously shown to characterize the curves of the second order.

(152.) First, let us seek the locus of the equation

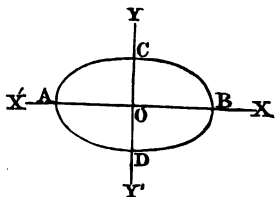
$$My^2 + Nx^2 = P \dots (1),$$

agreeing in form with the equation of the ellipse; M, N , and P being positive.

Assume a system of rectangular axes, XX', YY' , in reference to which the locus of (1) is to be constructed. Then since

$$\text{for, } y = 0 \text{ we have } x = \pm \sqrt{\frac{P}{N}}$$

$$\text{and for } x = 0 \text{ we have } y = \pm \sqrt{\frac{P}{M}}$$



If we make OB, OA each equal to $\sqrt{\frac{P}{N}}$, and OC, OD each equal to $\sqrt{\frac{P}{M}}$, the points A, B, C, D, thus determined, will be those in which the locus cuts the axes.

Let us now represent $\sqrt{\frac{P}{N}}$, that is, the abscissa OB by A; and $\sqrt{\frac{P}{M}}$, or the ordinate OC by B; then we shall have $N = \frac{P}{A^2}$, and $M = \frac{P}{B^2}$, and equation (1) will become, by substitution,

$$\frac{P}{B^2} y^2 = \frac{P}{A^2} x^2 = P, \text{ or } A^2 y^2 + B^2 x^2 = A^2 B^2 \dots (2.)$$

Let us now suppose that upon the lines AB, CD, as principal diameters, an ellipse is constructed; we know that this ellipse is analytically represented by equation (2); in other words, that it is the locus of equation (2). But equations (1) and (2) are identical; hence the same curve is the locus of equation (1).

We have here proceeded upon the supposition that $\sqrt{\frac{P}{N}} > \sqrt{\frac{P}{M}}$ or $M > N$; if, however, this be not the case, but $N > M$, then, putting X^2 for y^2 , and Y^2 for x^2 , in equation (1), it would have become $NY^2 + MX^2 = P$, which equation we should have shown, as above, to be the analytical representation of the ellipse constructed on the axes $2\sqrt{\frac{P}{M}}$ and $2\sqrt{\frac{P}{N}}$.

When $M = N$, equation (1) reduces to $y^2 + x^2 = \frac{P}{N}$, which represents a circle whose radius is $\sqrt{\frac{P}{N}}$.

To express the distance, c , of the centre of the ellipse, which is the locus of equation (1), from the focus, we have

$$A^2 = \frac{P}{N}, B^2 = \frac{P}{M} \therefore A^2 - B^2 = c^2 = \frac{P(M-N)}{MN}.$$

Secondly. Let us now suppose that the equation proposed is

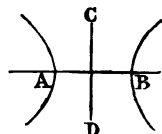
$$My^2 - Nx^2 = -P \dots (3)$$

agreeing in form with the equation of the hyperbola.

In this case

$$\text{for } y = 0 \text{ we have } x = \pm \sqrt{\frac{P}{N}}$$

$$\text{and for } x = 0 \text{ we have } y = \pm \sqrt{-\frac{P}{M}}$$



hence the locus of (3) cuts the axis of x in two points, equally distant from the origin, but it does not meet the axis of y .

Let $\sqrt{\frac{P}{N}}$ be represented by A , and $\sqrt{-\frac{P}{M}}$ by $\sqrt{-B^2}$; then N

$= \frac{P}{A^2}$, and $M = \frac{P}{B^2}$, and equation (3) becomes, by substitution, &c.

$$\frac{P}{B^2}y^2 - \frac{P}{A^2}x^2 = -P \text{ or } A^2y^2 - B^2x^2 = -A^2B^2 \dots (4.)$$

If, now, upon the principal diameters, $AB = 2\sqrt{\frac{P}{N}}$, or $2A$; and

$CD = 2\sqrt{\frac{P}{M}}$, or $2B$, an hyperbola be supposed to be constructed,

it will be analytically represented by the equation (4), that is, this curve will be the locus of equation (4); therefore, since equations (4) and (3) are identical, the hyperbola is also the locus of equation (3).

If, in equation (3), P had been positive, instead of negative, the locus would still have been an hyperbola; for then by putting X^2 for y^2 , and Y^2 for x^2 , the equation becomes, by changing the sides, $-P = NY^2 - MX^2$, the locus of which has just been shown to be an hyperbola.

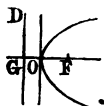
The distance, c , between the centre and focus of the hyperbola, which is the locus of equation (3), is found, as in the preceding case, to be

$$c = \pm \sqrt{\frac{P(M+N)}{MN}}.$$

Thirdly. Let the equation to be constructed be

$$y^2 = Qx \dots (5),$$

corresponding in form to the equation of the parabola. Then from the origin O of the given rectangular axes take two distances, OF , OG each equal to $\frac{1}{2}Q$; and having drawn the perpendicular, GD , if on the proposed axes a parabola be described, having F for its focus, and GD for its directrix, its equation will obviously be identical with equation (5); the locus of this equation is therefore a parabola.



(153.) It must here be remarked that the generality of the foregoing conclusions is not in the least diminished, because the axes to which the several loci are referred have been supposed rectangular. For, if they had been in each case oblique, we might, by employing the formulas at (40), have obtained the equation of the same locus for rectangular axes, after which we could, as in articles (50); (87), &c. have so determined the angles α and α' as to have preserved the form of the equation unaltered.

It must be further observed that if, in equation (1), P be supposed negative the locus will not be an ellipse, but an imaginary curve, since the general value for any ordinate is, in this case, the imaginary expression.

$$y = \sqrt{\left(-\frac{N}{M}x^2 - \frac{P}{M}\right)}.$$

This imaginary curve, as also a circle, and a point, all arising from equation (1), under different modifications of the constants, are called *varieties of the ellipse*.

By supposing N as well as P negative, in equation (1), this equation becomes identical with (3), characterizing an hyperbola. It will not, however, represent an hyperbola, if $P = 0$, but a system of two straight lines intersecting at the origin; for any ordinate of the locus will then be

$$y = \pm \sqrt{\frac{N}{M}x};$$

These straight lines are the asymptotes of the hyperbola (3) or (4) since

$$\pm \frac{B}{A} = \pm \sqrt{\frac{M}{N}}$$

and the reason that the equation, in this extreme case, is the representative of straight lines instead of a curve, is that that equation passes from the state of a single equation of the second order, into another which is equivalent to a system of *two* equations of the first order; for it becomes

$$My^2 - Nx^2 = (\sqrt{My} + \sqrt{Nx})(\sqrt{My} - \sqrt{Nx}) = 0,$$

and this is evidently equivalent to the two equations

$$\sqrt{My} + \sqrt{Nx} = 0$$

$$\sqrt{My} - \sqrt{Nx} = 0$$

which, as we have just seen, are the equations of the asymptotes, into which the curve (3) merges, when P is reduced to zero.

Thus the *varieties of the hyperbola* are an equilateral hyperbola, and a system of two intersecting straight lines.

The equation (5) of the parabola would seem to furnish no variety; for the change in the sign of Q merely changes the position of the curve from the right to the left of the axis of y .

As, however, Q diminishes, the ordinates to the same abscissa shorten; bringing the opposite points of the curve nearer together, till at last, when Q vanishes, these opposite points coalesce, and the curve degenerates into, and becomes confounded with, the axis of x :—this straight line is therefore a variety of the parabola. But a little modification of equation (5) will enable us to perceive that the parts of the curve above and below this axis will, in a particular state of the constants, each merge into a *distinct* straight line. The modification to which we allude is simply the removal of the origin O to a more advanced position on the axis of x ; or, which is the same thing, causing the vertex of the curve to recede towards the left. Calling the amount of this recession a the equation will then be (40)

$$y^2 = Q(x + a)$$

which, under the same value of Q , of course represents the same curve, whatever be a ; its vertex, however, being further from the origin the more a is increased. Let a be increased to $\frac{A}{Q}$, or infinity;

and let Q be at the same time decreased to 0. It is evident that as these continuous variations proceed, the vertex becomes more and more remote, y^2 gradually approaches to, and at length attains, the fixed value $y^2 = A$; and the curve merges into the system of parallel straight lines represented by the equation $y = \pm \sqrt{A}$. These varieties of the parabola will be deduced from other considerations presently.

(154.) It has been shown (46) that the equation of the ellipse, in terms of the parameter, the origin being at the vertex, is

$$y^2 = px - \frac{p}{2A} x^2.$$

If, in this equation, we suppose A to increase indefinitely, while p , or the value of $\frac{2B^2}{A}$, remains constant, on account of a suitable increase of B , it is plain that the coefficient $\frac{p}{2A}$ will continually diminish, and will at length vanish, when A becomes infinite, that is, the equation will then become

$$y^2 = px,$$

agreeing in form with equation (5), above, which belongs to a parabola. It follows, therefore, that the parabola may be considered as a species of the ellipse; since it is the form the ellipse takes, when the major diameter becomes infinite. Considering the parabola in this light, several properties of it established in sect. III., chap. iv. of the First Part, might have been deduced from the properties of the ellipse. Thus, for instance, since it was shown in (68) that the locus of the intersections of tangents to the ellipse, with the perpendiculars drawn to them from the focus, was the circumference of a circle described on the major diameter, we might have inferred that, when the centre of this circle became infinitely distant from the vertex, A , any finite portion of the circumference might be considered as a straight line; and have thence concluded that the locus becomes a straight line, when the ellipse becomes a parabola. As, however, this mode of deduction is liable to objection, and is moreover unnecessary, we have in no instance thought proper to resort to it. Many of the properties of the parabola demonstrated in that chapter have been established, by independent processes, in a manner much more simple than the corresponding properties of the ellipse; on this account, therefore, it would have been wrong to have made them depend upon these latter. The property here referred to is an illustration of this remark.

(155.) We now proceed to show that the locus of every indeterminate equation of the second degree, containing two variables,

can be no other than one of the curves already considered. The proof of this will be established, provided we can show that the general equation

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0 \dots (1)$$

may always be transformed into another, either of the form

$$My^2 + Nx^2 = P, \text{ or } y^2 = Qx,$$

by merely altering the axes to which the locus of (1) is referred.

It is obvious that we are at liberty to consider (1) as the equation of the locus, in reference to rectangular axes; since, if the axes were oblique, we should, by employing the formulas (4), p. 86 of the former Part, be able to transform the equation into another, referring the curve to rectangular axes; and, as this transformed equation would have the same form as the primitive, we may therefore consider equation (1) as the result.

1. *To remove the term containing the product of the variables.*

For x and y in equation (1) substitute the values

$$\left. \begin{aligned} x &= x \cos \alpha - y \sin \alpha \\ y &= x \sin \alpha + y \cos \alpha \end{aligned} \right\} \dots (2).$$

by means of which values we pass from one system of rectangular coordinates to another, having the same origin. The result of this substitution is

$A \cos^2 \alpha$	$y^2 + 2A \sin \alpha \cos \alpha$	$xy + A \sin^2 \alpha$	x^2
$-B \sin \alpha \cos \alpha$	$B \cos^2 \alpha$	$B \sin \alpha \cos \alpha$	
$C \sin^2 \alpha$	$-B \sin^2 \alpha$	$C \cos^2 \alpha$	
	$-2C \sin \alpha \cos \alpha$		

M 2

$$\left. \begin{array}{l} + D \cos \alpha \quad y + D \sin \alpha \quad x + F = 0 \\ - E \sin \alpha \quad \quad \quad E \cos \alpha \end{array} \right\} \dots (3).$$

Now the value of α is arbitrary; we may therefore assume it such, that the second term of the transformed equation may vanish; this value will be determined by the equation

$$2A \sin \alpha \cos \alpha + B \cos^2 \alpha - B \sin^2 \alpha + 2C \sin \alpha \cos \alpha = 0,$$

or

$$(A - C) 2 \sin \alpha \cos \alpha + B (\cos^2 \alpha - \sin^2 \alpha) = 0;$$

which, by substituting

$$\sin 2\alpha \text{ for } 2 \sin \alpha \cos \alpha, \text{ and } \cos 2\alpha \text{ for } \cos^2 \alpha - \sin^2 \alpha,$$

becomes

$$(A - C) \sin 2\alpha + B \cos 2\alpha = 0,$$

from which we obtain

$$\tan 2\alpha = -\frac{B}{A - C} \} \dots (4).$$

If, therefore, in the formulas (2), we give to the angle α a value such that the tangent of double that angle may be the number

$-\frac{B}{A - C}$, no term containing xy can appear in the transformed equation. This term, therefore, is removed, by changing the directions of the rectangular axes; and the transformed equation then takes the form

$$My^2 + Nx^2 + Ry + Sx + F = 0 \dots (5).$$

2. To remove the terms containing the first power of the variables.

For x and y in equation (5) substitute the values

$$x = a + x$$

$$y = b + y$$

By means of which the locus of (5) will become referred to new axes, parallel to the primitive; equation (5) will thus be transformed to

$$My^2 + Nx^2 + 2Mb \left| \begin{array}{c} y + 2Na \\ R \end{array} \right| x + Mb^2 + Na^2 + Rb + Sa + F = 0$$

in which equation, in order that the terms containing x and y may disappear, there must exist the conditions

$$2Mb + R = 0, \text{ and } 2Na + S = 0,$$

which give

$$b = -\frac{R}{2M}, \text{ and } a = -\frac{S}{2N}.$$

These values of a and b , therefore, reduce equation (5) to the form

$$My^2 + Nx^2 = P,$$

P being put for $-Mb^2 - Na^2 - Rb - Sa - F$.

(156.) We have here proceeded upon the supposition that neither of the terms My^2 , Nx^2 is absent from equation (5). If, however, one of these, as Nx^2 , is absent, that is, if $N = 0$, then the coefficient of x , in the transformed equation, will be simply S ; and conse-

quently the term containing the first power of x can in this case vanish only when $S = 0$, that is, when it is also absent from equation (5). If then this term be not absent from equation (5), neither can it be removed from the transformed equation when $N = 0$.

We can, however, in this case, remove the term which is independent of the variables; for when $N = 0$ this term is $Mb^2 + Rb + Sa + F$, and, in order to find what value must be given to the arbitrary quantity, a , that this expression may be 0, we must determine a from the condition

$$Mb^2 + Rb + Sa + F = 0,$$

which gives

$$a = -\frac{Mb^2 + Rb + F}{S}.$$

With this value of a , therefore, and the value $-\frac{R}{2M}$ for b , equation (5), in the case proposed, is reduced to the form

$$My^2 + Sx = 0, \text{ or } y^2 = Qx,$$

Q being put for $-\frac{S}{M}$.

If we had supposed $M = 0$, instead of $N = 0$, the resulting equation would have been $Nx^2 + Ry = 0$, agreeing with the preceding in form, when the axes are transposed.

We cannot suppose that, in equation (5), both $M = 0$ and $N = 0$ at the same time; or, which is the same thing, that the first three terms can vanish from equation (3). For, from inspecting the coefficients of these terms, it is obvious that the first and third cannot vanish, unless $A = -C$, and $B = 0$,* and upon this sup-

* These conditions will evidently destroy the first and third terms upon assuming the arbitrary angle α equal to 45° .

position the second term must remain, unless we moreover suppose that A and C are both 0, when equation (1) will be no longer of the second degree, but of the first, which is contrary to the hypothesis; so that the supposition of both M and N disappearing from equation (5) is inadmissible.

If both the first and second powers of one of the variables, as Sx and Nx^2 , are absent from equation (5), then the form of that equation is

$$My^2 + Ry + F = 0,$$

which is no longer an equation containing *two* variables, and represents not a curve, but a system of two straight lines parallel to the axis of x , for there are two constant values for each ordinate, viz.

$$y = -\frac{R}{M} \pm \frac{1}{2M} \sqrt{R^2 - 4FM}.$$

The two parallels characterized by this equation coincide, if $R^2 = 4MF$, and they become imaginary, if $4FM > R^2$.

This *variety* of the general equation, arising from the supposition that $N = 0$ and $S = 0$, in its transformed state (5), is a variety of the parabola, because when N is absent (5) evidently is of the same form as the equation of the parabola when referred to rectangular axes parallel to the principal axes of the curve.

(157.) We have now shown that every indeterminate equation of the second degree, containing two variables, may, by means of a double transformation of coordinates, be always reduced to one of the forms

$$My^2 + Nx^2 = P, \text{ or } y^2 = Qx,$$

except in the particular case, where the removal of xy by the first transformation takes away also the terms containing the first and second powers of one of the variables, in which case, the locus is a system of parallel straight lines. Hence the locus of (1) can be no

other curve but one of the three already discussed, that is, this locus must be either,

I. An ellipse, having for varieties a circle, a point, or an imaginary curve.

II. An hyperbola, having for varieties an equilateral hyperbola, or a system of two straight lines, intersecting at the origin.

III. Or a parabola, having for varieties a system of two parallel straight lines, a single straight line, or two imaginary straight lines.

As to the *varieties*, we have seen that those among them which are not *curves* arise from a change in the character of the equation by which it ceases to be an *indeterminate* equation of the second degree, and becomes either a compound of two equations of the first degree, or else a *determinate* equation of the second; and thus the proposition announced at the outset is fully established, viz. that every equation of the second order of whatever form, provided only it contain two variables must represent a conic section, if it represent any real line at all.

(158.) It is of importance to be able to ascertain readily, when any equation of the second degree is given, to which of the three curves it belongs. The following process will lead to a criterion for this purpose:

By adding and subtracting the values of M and N, and substituting in the first result, 1 for $\sin^2 \alpha + \cos^2 \alpha$, we have

$$\begin{aligned} M &= A \cos^2 \alpha - B \sin \alpha \cos \alpha + C \sin^2 \alpha \\ \frac{N}{M+N} &= \frac{A \sin^2 \alpha + B \sin \alpha \cos \alpha + C \cos^2 \alpha}{A + C} \\ M - N &= (A - C) (\cos^2 \alpha - \sin^2 \alpha) - B 2 \sin \alpha \cos \alpha \\ &= (A - C) \cos 2\alpha - B \sin 2\alpha. \end{aligned}$$

If, in this last equation, we substitute for $\cos 2\alpha$, $\sin 2\alpha$, their values in terms of $\tan 2\alpha = -\frac{B}{A-C}$, which, since

$$\cos = \frac{1}{\sqrt{1+\tan^2}}, \text{ and } \sin = \tan \cdot \cos$$

are

$$\cos 2a = \frac{A - C}{\sqrt{\{(A - C)^2 + B^2\}}}, \sin 2a = \frac{-B}{\sqrt{\{(A - C)^2 + B^2\}}}$$

we shall have

$$\begin{aligned} M - N &= \frac{(A - C)^2 + B^2}{\sqrt{\{(A - C)^2 + B^2\}}} \\ &= \sqrt{\{(A - C)^2 + B^2\}}; \end{aligned}$$

consequently

$$\begin{aligned} M &= \frac{1}{2}[(A + C) \pm \sqrt{\{(A - C)^2 + B^2\}}] \\ N &= \frac{1}{2}[(A + C) \mp \sqrt{\{(A - C)^2 + B^2\}}]. \end{aligned}$$

By multiplying these two expressions together, we have

$$M \cdot N = \frac{1}{4}[(A + C)^2 - \{(A - C)^2 + B^2\}] = \frac{1}{4}(4AC - B^2).$$

From this equation it follows that M and N must have the same sign, so long as $4AC > B^2$, that they must have different signs, when $4AC < B^2$, and that one of these coefficients must be 0, when $4AC = B^2$. Hence the general equation of the second degree (1) characterizes, when

$B^2 - 4AC < 0$, the ellipse, and its varieties;

$B^2 - 4AC > 0$, the hyperbola and its varieties;

$B^2 - 4AC = 0$, the parabola, and its varieties;

From the equation of $M + N = A + C$ it follows, that, if $A = -C$, then $M + N = 0$, that is, $M = -N$; the equation, therefore, denotes in this case, an equilateral hyperbola.

Since, when the general equation belongs to the parabola and its varieties, there must be $B^2 = 4AC$, the preceding expressions for

M and N become, in this case,

$$M = \frac{1}{2}[(A + C) \pm (A + C)] = A + C, \text{ or } 0,$$

$$N = \frac{1}{2}[(A + C) \mp (A + C)] = 0, \text{ or } A + C.$$

The upper sign of the quantity, $\pm \sqrt{(A - C)^2 + B^2}$, which occurs in the general expressions for M and N, is to be used when B is negative, and the lower sign when this coefficient is positive. For the sine of an angle being positive, whether the angle be acute or obtuse, it follows that the above quantity, which forms the denominator in the expression for $\sin 2\alpha$, above, must agree in sign with the numerator.

(159.) Having thus determined the coefficients, M, N, of the equation

$$My^2 + Nx^2 + Ry + Sx + F = 0,$$

we may thence obtain the values of R and S. We shall, in order to this, have to substitute for $\sin \alpha$, $\cos \alpha$, in the coefficients of x and y , in equation (3) their respective values

$$\cos \alpha = \sqrt{\frac{1 + \cos 2\alpha}{2}}, \sin \alpha = \sqrt{\frac{1 - \cos 2\alpha}{2}}$$

and, as $\cos 2\alpha$ has already been expressed in terms of the coefficients of the proposed equation, we shall thus obtain known values for all the coefficients of the above equation, which may then be simplified by art. (155) or (156).

When the locus of the proposed equation is not a parabola, that is, when $B^2 \not\geq 0$, it is plain, from the foregoing expressions for $\cos 2\alpha$, that the above values of $\cos \alpha$, $\sin \alpha$, will be rather complicated; much more so than when the locus is a parabola, or when $B^2 = 4AC$, since the expression for $\cos 2\alpha$ becomes then free from radicals. On this account it will be found more convenient, when

we have actually to construct the equation in the cases $B^2 - 4AC = \begin{matrix} < \\ > \end{matrix} 0$, first to remove the terms containing the first power of the variables, and afterwards to remove that containing their product. The two curves comprised in these cases are called *central curves*, to distinguish them from parabolas, which have no centre, their diameters being infinite. We shall now proceed to determine formulas for the construction of central curves, by reversing, as here recommended, the order of transformation before used.

Construction of Central Curves.

(160.) Resuming the general equation,

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0 \dots (1),$$

let us remove the origin of coordinates, by means of the formulas

$$x = a + x, y = b + y,$$

and the transformed equation becomes

$$Ay^2 + Bxy + Cx^2 + 2Ab \left| y + 2Ca \right| x + Ab^2 + Bab + Ca^2 + Db + Ea + F = 0.$$

$$\begin{array}{cc|c} Ba & Bb & \\ D & E & \end{array}$$

In order that the terms containing x and y may disappear from this equation, we must have the conditions

$$2Ab + Ba + D = 0 \dots (2)$$

$$2Ca + Bb + E = 0 \dots (3);$$

whence

$$a = \frac{2AE - BD}{B^2 - 4AC}, b = \frac{2CD - BE}{B^2 - 4AC}.$$

These values of a and b therefore reduce the transformed equation to the form

$$Ay^2 + Bxy + Cz^2 + F' = 0 \dots (4),$$

of which the first three coefficients are the same as those in the primitive equation; and where

$$F' = Ab^2 + Bab + Ca^2 + Db + Ea + F.$$

This expression for F' may be simplified by means of the conditions (2) and (3); for, multiplying the first by b , and the second by a , we have for their sum

$$2Ab^2 + 2Bab + 2Ca^2 + Db + Ea = 0$$

$$\therefore Ab^2 + Bab + Ca^2 = -\frac{Db + Ea}{2}$$

$$\therefore F' = F + \frac{Db + Ea}{2}.$$

The transformation just employed brings the origin of the axes to the centre of the curve; for equation (4) will remain unaltered, if for x we substitute $-x$, provided we at the same time change y into $-y$; so that, if (x', y') be one point in the curve, $(-x', -y')$ will always be another, and the line joining them will obviously pass through, and be bisected by, the origin; as, therefore, the origin bisects all the chords passing through it, it must be at the centre. Equation (4) is, therefore, the general equation of central curves, when the axes originate at the centre, and have any inclination whatever to each other. Had we known that the ellipse and hyperbola were the only curves coming under this denomination, the same

thing might have been inferred from the general equations of them in (50) and (87). If in the general equation (4) we put first $y = 0$, and then $x = 0$, the resulting values of x and y will be the lengths of those semi-diameters of the curve which are coincident with the axes of reference. The squares of these semi-diameters are therefore

$$x^2 = -\frac{F'}{C}, \quad y^2 = -\frac{F'}{A}$$

from which it appears that *to whatever axes a central curve be referred by its equation (1), the squares of the semi-diameters parallel to these axes will be to each other as A to C.*

Although equation (4) is entirely independent of the inclination of the axes, yet, for simplicity, we shall, as in the former mode of transformation, consider the axes as rectangular. To pass from these to the axes of the curve, we shall have to remove from equation (4) the term containing xy , by a transformation which has in (155) already been effected for the general equation, with which equation (4) agrees when $D = 0$, $E = 0$, and $F = F'$; so that the transformed equation (5), at p. 10, will here be

$$My^2 + Nx^2 + F' = 0;$$

hence, putting $P = -F'$, we have, finally,

$$My^2 + Nx^2 = P \dots (5).$$

(161.) We may now, for more convenient use, collect together the formulas to be employed, in order to transform an equation from the form (1) to the form (5), in those cases where $B^2 - 4AC < 0$, or $B^2 - 4AC > 0$.

I. PROPOSED EQUATION OF THE CURVE

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0.$$

1. *Formulas to be employed for removing the terms Dy, Ex.*

$$a = \frac{2AE - BD}{B^2 - 4AC}, \quad b = \frac{2CD - BE}{B^2 - 4AC}, \quad F' = F + \frac{Db + Ea}{2}.$$

II. RESULTING EQUATION,

The origin of the rectangular axes being at the centre,

$$Ay^2 + Bxy + Cx^2 + F' = 0.$$

2. *Formulas for the removal of the term Bxy, (see art. 158.)*

$$\tan 2\alpha = -\frac{B}{A - C}$$

$$M = \frac{1}{2}[(A + C) \pm \sqrt{\{(A - C)^2 + B^2\}}]$$

$$N = \frac{1}{2}[(A + C) \mp \sqrt{\{(A - C)^2 + B^2\}}]$$

$$P = -F'.$$

III. RESULTING EQUATION,

The axes being the principal diameters of the curve

$$My^2 + Nx^2 = P.$$

Since $P = -F'$ it follows from (153), that, when $F' = 0$, the locus will be two intersecting straight lines, if $B^2 - 4AC > 0$; and a point, if $B^2 - 4AC < 0$. In the case $B^2 - 4AC < 0$, the locus will be an imaginary curve, provided F' be positive. Hence we may always ascertain whether an equation of a central curve represents merely one of its varieties, by ascertaining the character of

F' by means of the given coefficients, and the conditions expressed in the first group of formulas above. These conditions, it will be remembered, are not dependent upon any particular inclination of the axes; and these axes are regarded as rectangular in the resulting equation (II.), only because the subsequent transformation is inapplicable to the case of oblique axes; the formulas having been deduced from the general investigation at (155).

EXAMPLES.

(162.) 1. Construct the locus of the equation

$$y^2 - 2xy + 3x^2 + 2y - 4x - 3 = 0.$$

Comparing this with the general equation (1), we find

$$A = 1, B = -2, C = 3, D = 2, E = -4, F = -3;$$

hence, substituting these values in the first class of formulas, above, we have

$$a = \frac{1}{2}, b = -\frac{1}{2}, F' = -\frac{3}{2};$$

therefore the equation of the curve, when the origin is removed to the centre, is

$$y^2 - 2xy + 3x^2 - \frac{3}{2} = 0.$$

By the second class of formulas we have

$$\tan 2a = -1, M = 2 + \sqrt{2}, N = 2 - \sqrt{2}, P = \frac{3}{2};$$

hence the equation of the curve, when related to its principal diameters, is

$$(2 + \sqrt{2})y^2 + (2 - \sqrt{2})x^2 = \frac{3}{2}.$$

To determine the values of the diameters $2A$ and $2B$, we have, by

supposing $y = 0$, in this equation,

$$x^2 = \frac{9}{2(2 - \sqrt{2})} = \frac{3}{2}(2 + \sqrt{2})$$

$$\therefore A = \frac{3}{2}\sqrt{2 + \sqrt{2}} = 2.7,$$

also for $x = 0$, we have

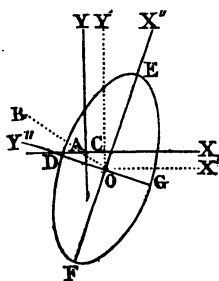
$$y^2 = \frac{9}{2(2 + \sqrt{2})} = \frac{3}{2}(2 - \sqrt{2})$$

$$\therefore B = \frac{3}{2}\sqrt{2 - \sqrt{2}} = 1.1.$$

The construction of the curve is, therefore, as follows:

Let AX, AY, be the original axes, to which the curve is referred. Make $AC = \frac{1}{2}$, $CO = -\frac{1}{2}$, then O will be the centre of the curve, and OX' , OY' , parallel to the primitive axes, will be the axes to which the first transformed equation refers the curve.

From O draw the straight line OB, making with OX' an angle of which the tangent is -1 , that is, an angle of 135° . Bisect this angle by the line OX'' , then OX'' and the perpendicular to it, OY'' , will be the axes to which the second transformed equation refers the curve; therefore, taking on these axes OE, OF, each equal to 2.7 , and OD, OG, each equal to 1.1 ; the principal diameters of the ellipse will be determined, and thence the curve easily traced.*



2. Construct the locus of the equation

$$2y^2 - 2xy - x^2 + 4x - 10 = 0.$$

* The expression for the distance between the centre and focus is given at page 4.

Here

$$A = 2, B = -2, C = -1, D = 1, E = 4, F = -10,$$

and, since $B^2 - 4AC > 0$, the locus is an hyperbola; also

$$a = \frac{1}{2}, b = \frac{1}{2}, F' = -7;$$

hence, when the origin is removed to the centre, the equation is

$$2y^2 - 2xy - x^2 - 7 = 0.$$

Again,

$$\tan 2\alpha = \frac{1}{2}, M = \frac{1}{2} + \frac{1}{2}\sqrt{13}, N = \frac{1}{2} - \frac{1}{2}\sqrt{13}, P = 7,$$

therefore the equation of the curve, when referred to its principal diameters, is

$$\left(\frac{1}{2} + \frac{1}{2}\sqrt{13}\right)x^2 + \left(\frac{1}{2} - \frac{1}{2}\sqrt{13}\right)y^2 = 7,$$

or

$$(1 + \sqrt{13})x^2 + (1 - \sqrt{13})y^2 = 7.$$

When $y = 0$,

$$x^2 = \frac{27}{2(1 + \sqrt{13})} = \frac{1}{2}(\sqrt{13} - 1) = 2.93 = A^2;$$

when $x = 0$,

$$y^2 = \frac{27}{2(1 - \sqrt{13})} = -\frac{1}{2}(\sqrt{13} + 1) = -5.18 = B^2.$$

Hence, as in the preceding example, there are given the axes of the curve, to construct it.

3. Determine the axes of the curve of which the equation is

$$2y^2 - 4xy + 5x^2 - 3x = 0.$$

Ans. The curve is an ellipse, whose axes are $\sqrt{3}$ and $\frac{1}{2}\sqrt{2}$,

4. Required the axes of the curve which is the locus of the equation

$$5y^2 + 2xy + 5x^2 + 2y - 2x - \frac{1}{2} = 0.$$

Ans. The curve is an ellipse, whose axes are $2\sqrt{\frac{1}{2}}$ and $2\sqrt{\frac{1}{2}}$.

5. Required the axes of the curve which is the locus of equation

$$y^2 - 6xy + x^2 + 2y - 6x + 5 = 0.$$

Ans. The curve is an hyperbola, whose axes are $2\sqrt{2}$ and 2 .

6. What is the locus of the equation

$$y^2 - 6xy + x^2 + 2y - 6x + 1 = 0?$$

Ans. Two straight lines, characterized by the equation $y = x\sqrt{\frac{1}{2}}$.

7. What is the geometrical representation of the equation

$$y^2 - 4xy + 5x^2 + 2x + 1 = 0?$$

Ans. A point $(-1, -2)$.

8. What is the locus of the equation

$$2x^2 + 2y^2 - 3x + 4y - 1 = 0?$$

Ans. A circle whose radius is $\frac{1}{2}\sqrt{13}$.

9. What is the locus of the equation

$$y^2 - 2xy + 2x^2 - 2x + 4 = 0?$$

Ans. An imaginary curve.

10. What is the locus of the equation

$$y^2 + 2xy - 2x^2 - 4y - x + 10 = 0?$$

Ans. An hyperbola, in which the second axis is taken for the axis of abscissas. The axes are 1.7 and 2.2 .

11. What is the geometrical representation of the equation

$$2y^2 + 3x^2 - 3x - 2y + 2 = 0?$$

Ans. The equation has no geometrical representation.

12. What is the locus of the equation

$$3y^2 + 6x^2 - 24x + 6 = 0,$$

the axes of reference being oblique?

Ans. An ellipse in which the semi-conjugates parallel to the axes of reference are $\sqrt{3}$ and $\sqrt{6}$.

(163.) Before we proceed to the construction of parabolas, we shall remark, that if, in the equations of condition (2), (3), (*art.* 160), we substitute for the constants a, b , the variables x, y , the equations

$$2Ay + Bx + D = 0 \dots (1)$$

$$2Cx + By + E = 0 \dots (2)$$

will characterize two straight lines, passing through the centre of the locus; as is evident, since a, b , the coordinates of this centre, satisfy both equations. Were we to suppose these lines to be parallel, or the centre of the locus to be infinitely distant, as in the parabola, we could infer, from the equations (1), (2), that in the equation of the locus there must be $B^2 - 4AC = 0$. For these equations give

$$y = -\frac{Bx + D}{2A}, \text{ and } y = -\frac{2Cx + E}{B} \dots (3);$$

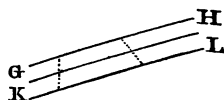
and since, when the lines are parallel, the difference of the ordinates corresponding to every abscissa must be constant, we have, by reducing these expressions to the same denominator,

$$B^2x + BD - 4ACx - 2AE = \text{constant}$$

$$\text{or } (B^2 - 4AC)x + BD - 2AE = \text{constant},$$

which can only happen when $B^2 - 4AC = 0$.

~ If the lines (1) and (2) coincide, we must conclude that the locus has an infinite number of centres, and all situated in the line (1) or (2). The locus, therefore, can be no other than a system of parallel straight lines, as GH, KL, equally distant from the line through the centres; for then every chord of the locus must be bisected by this line. We already know (156) that this locus is a variety of the parabola. Equations (1) and (2) will also show this to be the case; and will moreover furnish an additional criterion, whereby we may readily ascertain, by inspecting the coefficients of the proposed equation, when that equation characterizes a system of parallels, and when it does not. For, since, in this case, equations (1) and (2) represent the same line, we have, equation (3),



$$\frac{Bx + D}{2A} = \frac{2Cx + E}{B}$$

$$\therefore B^2x + BD = 4ACx + 2AE,$$

whatever be the value of x ; consequently (*Alg.* p. 164)

$$B^2 = 4AC, \text{ and } BD = 2AE;$$

so that, when the indeterminate equation of the second degree represents a system of parallels, there must exist among its coefficients the conditions

$$B^2 - 4AC = 0, \text{ and } BD - 2AE = 0.$$

These lines may be at once determined from the given equation; for, being parallel, the coefficient of x must necessarily be the same in the equation of each; that is, these equations will be of the form

$$y + px + q = 0, \text{ and } y + px + r = 0 \dots (3)$$

so that the proposed equation, after having freed y^2 from its coefficient, may always, in the case we are considering, be decomposed into two factors of this form; where p , the coefficient of x , in each must be equal to half the coefficient of xy , in the proposed equation, after this has been divided by A , the coefficient of the

first term: hence $p = \frac{B}{2A}$. With regard to q and r , it is plain

that their sum must be equal to $\frac{D}{A}$, the coefficient of y , in the pro-

posed, and their product must make the last term $\frac{F}{A}$. Having

thus the sum and the product of q and r , we shall get their difference, by subtracting four times the product from the square of the sum, and extracting the square root; that is,

$$q - r = \sqrt{\left(\frac{D^2}{A^2} - \frac{4F}{A}\right)} = \frac{1}{A} \sqrt{D^2 - 4AF};$$

therefore, adding the half difference to the half sum, we have, for the greater,

$$q = \frac{1}{2A} (D + \sqrt{D^2 - 4AF});$$

and, by subtracting the same, we get the less. Now it is plain, from this expression, that, if $D^2 = 4AF$, then $q = r$; hence the two equations (3) become, in this case, identical; and the parallels, therefore, coincide, and become a single straight line. If $D^2 < 4AF$, then the values of q and r become imaginary, so that, in this case, the locus is two imaginary lines.

From what has now been said, we may conclude that the equation

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0$$

represents a system of parallel lines when

$$B^2 - 4AC = 0, \text{ and } BD - 2AE = 0,$$

these coincide, and form but one straight line when in conjunction also with these conditions, we have

$$D^2 - 4AF = 0,$$

and they become imaginary when, instead of this,

$$D^2 - 4AF < 0.$$

When the lines are real, their equations (3) are

$$y + \frac{B}{2A}x + \frac{1}{2A}(D + \sqrt{D^2 - 4AF}) = 0,$$

and

$$y + \frac{B}{2A}x + \frac{1}{2A}(D - \sqrt{D^2 - 4AF}) = 0.$$

When they coincide, the equation is

$$y + \frac{B}{2A}x + \frac{D}{2A} = 0.$$

(164.) As, in the case we are discussing, the factors of the original equation consist of the sum and difference of the same two quantities, their product must be the difference of the squares of these quantities; hence, when the proposed equation represents two straight lines, it must consist of the difference of two squares, or at least of one square, minus a number. On the contrary, when the lines represented are imaginary, the equation must consist of the sum of two squares, or at least of one square and a number. And the equation will be a perfect square when only one straight line is represented. Hence we may frequently discover at a glance when the equation denotes a variety of the parabola, without even

trying whether $BD - 2AE = 0$. Thus we see at once that the equation

$$y^2 - 2xy + x^2 - 1 = 0$$

is the difference of two squares, viz. $(y - x)^2$, and 1; the locus of it is therefore two parallel straight lines, the equations of which are

$$y - x + 1 = 0, \text{ and } y - x - 1 = 0.$$

Also, the equation

$$y^2 - 4xy + 4x^2 + 9 = 0$$

is immediately seen to consist of the two squares $(y - 2x)^2$, and 9, therefore the locus is imaginary.

In like manner, since the equation

$$y^2 + 4xy + 4x^2 + 2y + 4x + 1 = 0$$

is obviously a perfect square, viz. $(y + 2x + 1)^2$, its locus is a straight line, the equation of which is

$$y + 2x + 1 = 0.$$

Let, now, the equation

$$y^2 + 6xy + 9x^2 - 2y - 6x - 15 = 0$$

be proposed, which is a variety of the parabola, because $B^2 - 4AC = 0$; and since, moreover, $BD - 2AE = 0$, this variety is a system of parallels, of which the equations are

$$y + 3x + 3 = 0, \text{ and } y + 3x - 5 = 0.$$

Lastly, let the equation be

$$y^2 - 4xy + 4x^2 + 2y - 4x + 4 = 0,$$

the coefficients of which furnish, besides the conditions above, the

relation $D^2 - 4AF < 0$, therefore the locus is imaginary. This equation, being the same as

$$(y - 2x + 1)^2 + 3 = 0,$$

consists of a perfect square and a number.

(165.) We shall now proceed to furnish formulas for the construction of parabolas, as we have already done for the central curves. In the present case, our object will be first to remove xy from the equation, and afterwards to remove the term containing the first power of one of the variables, and the absolute number. The first transformation, as we have already seen (155), brings the equation to the form

$$My^2 + Ry + Sx + F = 0, \text{ or } Nx^2 + Ry + Sx + F = 0,$$

where

$$R = D \cos \alpha - E \sin \alpha, \quad S = D \sin \alpha + E \cos \alpha.$$

Now, because

$$\cos \alpha = \sqrt{\frac{1 + \cos 2\alpha}{2}}, \text{ and } \sin \alpha = \sqrt{\frac{1 - \cos 2\alpha}{2}}$$

and since, also, the expression given at (158) for the $\cos 2\alpha$ becomes, when $B^2 = 4AC$,

$$\cos 2\alpha = \frac{A - C}{A + C}, \text{ or } \frac{A - C}{-(A + C)}$$

accordingly, as B is negative or positive, we have, by substitution, the following expressions for R and S , viz. when B is negative,

$$R = \frac{D\sqrt{A} - E\sqrt{C}}{\sqrt{A + C}}, \quad S = \frac{D\sqrt{C} + E\sqrt{A}}{\sqrt{A + C}}$$

and when B is positive,

$$R = \frac{D\sqrt{C} - E\sqrt{A}}{\sqrt{A+C}}, \quad S = \frac{D\sqrt{A} + E\sqrt{C}}{\sqrt{A+C}}$$

(166.) The values of M and N have already been determined (158), as also the values of a and b , employed in the second transformation (155); hence, collecting these formulas together, we have

I. PROPOSED EQUATION OF THE' CURVE,

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0.$$

Formulas to be employed for removing the Terms containing the Product of the Variables, and the Square of one of them.

1. When B is negative,

$$\tan 2\alpha = -\frac{B}{A-C}, \quad M = A + C, \quad N = 0$$

$$R = \frac{D\sqrt{A} - E\sqrt{C}}{\sqrt{A+C}}, \quad S = \frac{D\sqrt{C} + E\sqrt{A}}{\sqrt{A+C}}$$

II. RESULTING EQUATION,

The rectangular axes being parallel to the axes of the curve,

$$My^2 + Ry + Sx + F = 0.$$

Formulas for the removal of the terms Ry and F (see p. 12).

$$b = -\frac{R}{2M}, \quad a = -\frac{Mb^2 + Rb + F}{S} = \frac{R^2 - 4MF}{4MS}$$

III. RESULTING EQUATION,

The axes being those of the curve,

$$My^2 + Sx = 0.$$

2. When B is *positive*,

The first class of formulas is

$$\tan 2\alpha = -\frac{B}{A-C}, \quad N=A+C, \quad M=0$$

$$R = \frac{D\sqrt{C-E}\sqrt{A}}{\sqrt{(A+C)}}, \quad S = \frac{D\sqrt{A+E}\sqrt{C}}{\sqrt{(A+C)}}$$

and the resulting equation

$$Nx^2 + Ry + Sx + F = 0,$$

The second class of formulas is

$$a = -\frac{S}{2N}, \quad b = \frac{S^2 - 4NF}{4NR}$$

and the final equation is $Nx^2 + Ry = 0$.

EXAMPLES.

1. Construct the curve, which is the locus of the equation

$$y^2 - 4xy + 4x^2 + 2y - 7x - 1 = 0.$$

Here $A = 1$, $B = -4$, $C = 4$, $D = 2$, $E = -7$, $F = -1$; and, as B is negative, we must employ the first collection of formulas, which give

$$\tan 2\alpha = -\frac{1}{2}, M=5, N=0, R=\frac{1}{2}\sqrt{5}, S=-\frac{1}{2}\sqrt{5}.$$

Hence, when the axes are parallel to those of the curve, the equation becomes

$$5y^2 + \frac{1}{2}\sqrt{5}y - \frac{1}{2}\sqrt{5}x - 1 = 0.$$

Again, formulas II. give

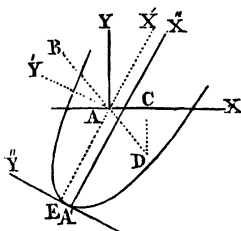
$$b = -\frac{1}{2}\sqrt{5}, a = -\frac{1}{2}\sqrt{5};$$

these, therefore, are the ordinate and abscissa of the principal vertex of the curve, the equation to which, in reference to the axes of the curve is

$$5y^2 - \frac{1}{2}\sqrt{5}x = 0, \text{ or } y^2 = \frac{1}{25}\sqrt{5}x.$$

The construction of this curve is therefore as follows:

Let AX, AY be the primitive rectangular axes. On the former take AC = 1, and make the perpendicular, CD = $-\frac{1}{2}$. Draw DAB, and bisect the angle BAX = 2α by the line AX'; then the rectangular axes, AX', AY', are those to which the first-transformed equation refers the curve.



Again, take AE = $-\frac{1}{2}\sqrt{5}$, and the perpendicular EA' = $-\frac{1}{2}\sqrt{5}$; then the axes A'X'', A'Y'', parallel to the former, will be those of the curve. Having thus the axes and the parameter $\frac{1}{25}\sqrt{5}$, the focus and directrix are readily determined, and thence the curve constructed.

2. Construct the locus of the equation

$$y^2 + 2xy + x^2 - 6y + 9 = 0.$$

$$\text{Here } A=1, B=2, C=1, D=-6, E=0, F=9,$$

and, as B is positive, the second collection of formulas must be

34 LOCI OF EQUATIONS OF THE SECOND DEGREE.

used, which give

$$\tan 2\alpha = \frac{-2}{0}, \quad M=0, \quad N=2, \quad R=-3\sqrt{2}, \quad S=-3\sqrt{2}.$$

Hence the equation of the curve, when the axes are parallel to those of the curve, is

$$2x^2 - 3\sqrt{2}y - 3\sqrt{2}x + 9 = 0;$$

and when they coincide with the axes of the curve, the coordinates of whose origin are

$$a = \frac{3}{2}\sqrt{2}, \quad b = \frac{3}{2}\sqrt{2},$$

the equation is

$$2x^2 - 3\sqrt{2}y = 0, \quad \text{or} \quad x^2 = \frac{3}{2}\sqrt{2}y.$$

Hence, as in the preceding example, the curve may be constructed. In the first transformation of axes, since $\tan 2\alpha$ is infinite, 2α is a right angle; so that in this transformation the new axis of x will be 45° below the primitive.

3. The equation of a parabola being

$$y^2 - 4xy + 4x^2 - 8y + 3x - 2 = 0,$$

what will it become when the curve is referred to its axes?

$$\text{Ans. } y^2 = 13\sqrt{5}x.$$

4. Required the principal parameter of the parabola whose equation is

$$4y^2 - 4xy + x^2 - 2y - 4x + 10 = 0.$$

$$\text{Ans. } p = \sqrt{\frac{4}{5}}.$$

5. What is the principal parameter of the parabola represented by the equation

$$y^2 - 2xy + x^2 - 3y = 0?$$

$$\text{Ans } p = \frac{3}{4\sqrt{2}}.$$

(167.) The student must bear in mind, that the various formulas given in this chapter for the construction of lines of the second order, apply only when the different equations refer the curves to rectangular axes, which, indeed, are in most cases employed. With regard, however, to the *varieties* of the three curves, the tests by which they may be discovered, and the formulas for their construction, apply generally, because in discussing these varieties, we have considered the axes to have any inclination whatever, and because, moreover, the criteria (158), by which the three classes of curves are distinguished, apply for every inclination of axes, as we are about to show in the following chapter, which has for its object the determination of the locus of the general equation when the axes are oblique.

CHAPTER II.

DISCUSSION OF THE GENERAL EQUATION

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0,$$

By the separation of the variables.

(168.) This equation may be put under the form

$$y^2 + \frac{Bx+D}{A}y + \frac{C}{A}x^2 + \frac{E}{A}x + \frac{F}{A} = 0,$$

which, solved as a quadratic, gives for y the expression

$$y = -\frac{Bx+D}{2A} \pm$$

$$\frac{1}{2A} \sqrt{\{(B^2 - 4AC)x^2 + 2(BD - 2AE)x + D^2 - 4AF\}} \dots (1).$$

In like manner, we have for x , in terms of y , the expression

$$x = -\frac{By + E}{2C} \pm \frac{1}{2C} \sqrt{\{(B^2 - 4AC)y^2 + 2(BE - 2CD)y + E^2 - 4CF\}} \dots (2).$$

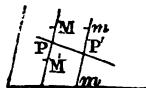
Either of these expressions will furnish an indefinite number of points in the locus, when this is not imaginary, since the first will give the ordinates corresponding to any assumed abscissa, and the second will give the abscissas corresponding to any assumed ordinate. If we wish to determine points in the curve from the equation (1), we must first, for any assumed abscissa, x , draw an

ordinate equal to $-\frac{Bx + D}{2A}$, determining some

point, P ; then, if this ordinate be prolonged, and the distances PM , PM' be taken thereon, each equal to the line represented by the remaining part of the expression for y , two points of the locus will thus be determined. P , therefore, is the middle of the chord MM' . The same construction for another value of x will determine another point, P' , and two new points, m , m' , of the curve, which will, as before, be equally distant from P' . Hence, calling the variable coordinates of these points P , P' , &c. X , Y , since we must always have

$$Y = -\frac{BX + D}{2A},$$

it follows that the locus of these points is a straight line, which, because it bisects all the chords in the curve drawn parallel to the axis of y , is called a diameter of the curve. Similar reasoning applied to the expression (2) will show that the straight line represented by the equation



$$X = -\frac{BY + E}{2C}$$

is a diameter, bisecting the chords drawn parallel to the axis of x . These diameters are obviously the same as those represented at (163).

Having thus the equation of two diameters, we can always readily find the centre of any locus of the second order, to whatever axes it be referred; for, representing the coordinates of the centre by a, b , we shall have, by substituting these for the coordinates in each of the preceding equations, and solving them, as at (160), the values

$$a = \frac{2AE - BD}{B^2 - 4AC}, b = \frac{2CD - BE}{B^2 - 4AC}.$$

(169.) From these remarks it appears that the nature of the curve depends upon the irrational part of the expression (1), or (2); and that it cannot exist when this irrational part becomes 0, or imaginary, for every value of the variable it contains. Let us examine the circumstances under which these irrational expressions can become real, imaginary, or nothing.

We shall first take the expression (1), and shall suppose that the quantity under the radical is decomposed into two factors, each containing x ; in other words, we shall suppose the solution of the equation

$$x^2 + \frac{2(BD - 2AE)}{(B^2 - 4AC)}x + \frac{D^2 - 4AF}{(B^2 - 4AC)} = 0 \dots (3)$$

to be effected, and that the resulting values of x are $x = \beta$ and $x = \beta'$; then we know (*Theory of Equations*, p. 9) that the multiplication of the factors $(x - \beta)$, $(x - \beta')$ will produce this equation; and consequently the quantity under the radical (1) will be

$$(B^2 - 4AC)(x - \beta)(x - \beta') \dots (4).$$

If, however, $B^2 - 4AC = 0$, then the expression under the radical

will have only one factor containing x , discoverable by solving the simple equation

$$x + \frac{D^2 - 4AF}{2(BD - 2AE)} = 0;$$

so that, putting δ for the value of x , in this equation, the expression under the radical will be

$$2(BD - 2AE)(x - \delta).$$

The form (4) therefore only exists when $B^2 - 4AC < 0$, or $B^2 - 4AC > 0$; let us examine the expression in the first case, viz.

$$(170.) \quad \text{When } B^2 - 4AC < 0.$$

There are three circumstances to consider in this case:

1. When the roots β, β' , are real and unequal.
2. When the roots are real and equal.
- 3, and lastly. When they are imaginary.

Suppose, first, that the roots are real and unequal, β being greater than β' , then (*Theory of Equations*, prop. 5, chap. iii.) if in the expression

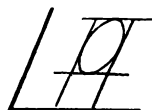
$$(B^2 - 4AC)(x - \beta)(x - \beta')$$

any quantity greater than β , or less than β' , be substituted for x , the product $(x - \beta)(x - \beta')$ will be positive; and since by hypothesis, $B^2 - 4AC$ is negative, the whole expression will be negative, and therefore for all such values of x , the expression for y will be imaginary. But, if we substitute for x any value between β and β' , then the product $(x - \beta)(x - \beta')$ will be negative, and consequently the expression (4) will be positive; for all such values of x , therefore, there correspond real values of y .

From this discussion it follows, that, under the conditions we have supposed, the curve always exists; and that it is comprised between, or *limited* by, two parallels to the axis of ordinates, drawn at the respective distances of β' and β from the origin; for between

these parallels all the values of x which give possible values for y are comprehended.

By applying precisely similar reasoning to the expression (2), it would result that the curve is also limited by two parallels to the axis of x ; as, therefore, these parallels meet the former, and form a parallelogram, circumscribing the curve, it follows that the curve must be limited in all directions, as in the annexed diagram. The curve, therefore, must necessarily be an ellipse.



Suppose, secondly, that the roots β, β' are real and equal; then the expression (4) is

$$(B^2 - 4AC)(x - \beta)(x - \beta);$$

where, since it is impossible to substitute any value for x between the roots β and β , it is, by the preceding reasoning, also impossible to render the expression for y real by any substitution for x , except in the single case $x = \beta$, which renders the irrational part of the expression 0, and reduces the value of y to

$$y = -\frac{B\beta + D}{2A}.$$

Hence, when the roots β, β' are equal, the curve is reduced to a point, of which the coordinates are

$$\beta, \text{ and } -\frac{B\beta + D}{2A}.$$

If, lastly, the roots be imaginary, then whatever value we substitute for x , in the equation containing them, the result will be positive (*Theory of Equations, art. 15*); hence every value of y will be imaginary, so that, in this case, the curve cannot exist.

We may now therefore infer, that when, in the general equation, $B^2 - 4AC < 0$, whatever be the inclination of the axes, the locus is an ellipse, if the roots of the irrational part of the expression for y be

real and unequal; it is merely a point, if these roots be equal; and it is imaginary, if the roots be so.

(171.) Let us now discuss the equation upon the second hypothesis, viz.

$$\text{When } B^2 - 4AC > 0.$$

Resuming the expression

$$(B^2 - 4AC) (x - \beta) (x - \beta'),$$

and reasoning as before, in reference to the roots β and β' , we find that here, when these roots are real, and β greater than β' , every value of x greater than β , or less than β' , will, because $B^2 - 4AC$ is positive, render the expression for y real; while, on the contrary, every value comprised between the limits β and β' will render the expression for y imaginary. As therefore, without these limits, x may increase indefinitely, both positively and negatively, it follows that the curve must consist of two infinite detached branches, proceeding in opposite directions, and separated from each other by the distance between two parallels to the axis of y , of which the abscissas are respectively β and β' ; for within these limits there exists no possible value of y .

This curve, therefore, is the hyperbola.

If the roots β, β' are equal, the expression above is

$$(B^2 - 4AC) (x - \beta)^2;$$

and hence the value of y becomes

$$y = -\frac{Bx + D}{2A} + \frac{\sqrt{B^2 - 4AC}}{2A} (x - \beta),$$

or

$$y = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} x - \frac{D \pm \beta \sqrt{B^2 - 4AC}}{2A}$$

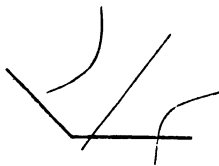


hence the locus is a *system of two straight lines*, which intersect, since the coefficient of x is not the same in both.

When the roots β, β' are imaginary, then, since every value given to x , in the equation containing them, gives a positive result, the whole expression under the radical will be positive, and, therefore, the value of y will be always real. As, therefore, x may take any value from 0 to infinity, in both directions, it follows that the curve is unlimited in both directions. It moreover consists of two distinct branches; for, as each double ordinate, or chord drawn parallel to the axis of y , is bisected by the diameter whose equation is

$$y = -\frac{Bx + D}{2A}$$

one half of the curve must be situated entirely below this line, and the other half above it; neither can have a point in common with this diameter, because the irrational part of the value of y can never vanish; hence the curve must be an hyperbola.



(172.) There is a particular form of the general equation which ought here to be noticed:—it is that where the squares of the variables are absent, when the equation becomes

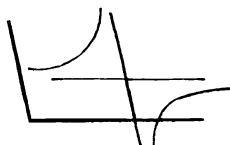
$$Bxy + Dy + Ex + F = 0 \dots (5),$$

which gives for y the expression

$$y = -\frac{Ex + F}{Bx + D}$$

and, as this value of y will always be real, whatever be the value of x , it follows that the curve extends indefinitely in opposite directions. As each value of x furnishes but one value of y , each ordinate meets the curve in but one point.

If the value $-\frac{D}{B}$ be given to x , the corresponding value of y will be infinite; that is, if a parallel to the axis of y be drawn at the distance of $-\frac{D}{B}$ from the origin, it



will never meet the curve; but, as every parallel drawn on either side of this must necessarily meet the curve, because no abscissa but $x = -\frac{D}{B}$ can render the ordinate infinite, it follows that the curve consists of two distinct branches, separated by the parallel whose abscissa is $-\frac{D}{B}$. The curve, therefore, is an hyperbola; and the parallel, whose abscissa is $-\frac{D}{B}$, is obviously one of the asymptotes, as this parallel has been seen to be the only one which does not meet the curve.

By solving the equation (5), with regard to x we have

$$x = -\frac{Dy + F}{By + E}$$

in which expression $-\frac{E}{B}$ is the only value that can be given to y , that will render x infinite; hence we infer here, that a parallel to the axis of x , of which the ordinate is $-\frac{E}{B}$, is the other asymptote of the curve. Hence equation (5) represents an hyperbola whose asymptotes are parallel to the axes of coordinates; the coordinates (x', y') of the point of intersection of the asymptotes being

$$x' = -\frac{D}{B}, y' = -\frac{E}{B} \dots (6).$$

The asymptotes are therefore easily determined when the equation of the hyperbola takes the form (5).

If the term Ax^2 had appeared in the equation (5), the same reasoning with regard to the expression for y would apply, inasmuch as the denominator of that expression would remain the same, and would therefore vanish for the same value of x ; so that

then also the parallel to the axis of y , of which the abscissa is $-\frac{D}{B}$, is an asymptote.

If Cy^2 appear in the equation, instead of Ax^2 , then, reasoning as above on the expression for x , we find that a parallel to the axis of x , of which the ordinate is $-\frac{E}{B}$ is an asymptote.

If both $A = 0$ and $D = 0$, the axis of y coincides with an asymptote. If both $C = 0$ and $E = 0$, the axis of x coincides with an asymptote. If both $D = 0$ and $E = 0$, the origin coincides with the intersection of the asymptotes, even though the squares of both variables are present, as is plain from the expressions for a and b , at page 37; but when the squares of the variables are absent, both axes coincide with the asymptotes, their equations being (6), and the equation (5) takes the form $Bxy + F = 0$.

To determine the asymptotes from the general equation, let us actually extract the root of the expression under the radical, in the general value of the ordinate (1): we shall find this root to be of the form

$$x\sqrt{B^2-4AC} + \frac{BD-2AE}{\sqrt{B^2-4AC}} + \frac{K}{x} + \frac{K'}{x^2} + \&c.$$

therefore

$$y = -\frac{Bx + D}{2A} \pm \frac{\sqrt{B^2-4AC}}{2A}x + \frac{BD-2AE}{2A\sqrt{B^2-4AC}} + \frac{K}{2Ax} + \frac{K'}{2Ax^2} + \&c.$$

Now it is here obvious that as x increases the term $\frac{K}{2Ax}$, and all that follow will diminish, while those that precede will increase; and to these first terms the expression is finally reduced, when x becomes infinite. Hence the curve continually approaches the two straight lines denoted by

$$Y = -\frac{Bx + D}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A} \left(x + \frac{BD - 2AE}{B^2 - 4AC} \right)$$

these lines are therefore the asymptotes to the curve.

Comparing this equation with the equation at p. 40, which represents the locus when it becomes a system of two straight lines, we shall find them to be identical. For, as that equation takes place only when the roots β, β' are equal, it follows that then β must be equal to minus half the coefficient of x , in the equation (3), at p. 37, which contains them; that is, we must have

$$\beta = -\frac{BD - 2AE}{B^2 - 4AC}$$

which value of β renders the equation (p. 40) identical with that above for the asymptotes. We may therefore say that, when the equation represents a system of straight lines, the hyperbola degenerates into its asymptotes.

(173.) We already know that the asymptotes intersect at the centre, this is also readily ascertained from their equation above; for since at their intersection the two values of Y coincide, the difference of the two expressions for it above will be zero; whence we have for the abscissa, a , of the intersection the value

$$a = \frac{2AE - BD}{B^2 - 4AC}$$

which (168) is the abscissa of the centre; and the corresponding value of the ordinate is

$$b = -\frac{Bx + D}{2A} = \frac{2CD - BE}{B^2 - 4AC}$$

which (168) is the ordinate of the centre. Hence, if we wish to construct the asymptotes, when the equation of the hyperbola appears under the general form, we shall have first to determine the centre from these formulas, and then to draw through this point two straight lines inclined to the axis of x , at angles α , α' , whose tangents* are respectively

$$\tan \alpha = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \text{ and } \tan \alpha' = \frac{-B - \sqrt{B^2 - 4AC}}{2A}.$$

The product of these two tangents is

$$\tan \alpha \cdot \tan \alpha' = \frac{4AC}{4A^2} = \frac{C}{A}$$

which, when $C = -A$, becomes

$$\tan \alpha \cdot \tan \alpha' = -1,$$

an equation which indicates (when the axes of reference are rectangular) that the asymptotes are perpendicular to each other (12). Hence, if, in the general equation, $B^2 - 4AC > 0$, and $C = -A$, when the locus is referred to rectangular axes, we may conclude that the equation represents an equilateral hyperbola.

It must be here remarked, that, when $A = 0$, the preceding expression for $\tan \alpha$ becomes $\frac{0}{0}$, which is not a definite result; but,

* We are here supposing the axes to be rectangular; if they are oblique, then for \tan substitute ratio of the sines.

by multiplying numerator and denominator by $B + \sqrt{B^2 - 4AC}$, it reduces to

$$\tan \alpha = \frac{-2C}{B + \sqrt{B^2 - 4AC}} = -\frac{C}{B}, \text{ when } A = 0.*$$

(174.) We shall now examine the general equation upon the third hypothesis, viz.

$$\text{When } B^2 - 4AC = 0.$$

Under this condition, the general expression for any ordinate of the locus is

$$y = -\frac{Bx + D}{2A} \pm \frac{1}{2A} \sqrt{\{2(BD - 2AE)x + D^2 - 4AF\}}.$$

If we put

$$\beta \text{ for } -\frac{D^2 - 4AF}{2(BD - 2AE)}$$

* A fraction which takes the form $\frac{0}{0}$, for a particular value of one of the arbitrary quantities which enter it, can do so only in consequence of a factor, common to both numerator and denominator, vanishing for the particular value in question. The common factor, in the instance above, can be no other than A ; as is obvious from the form of the denominator: and that the numerator involves this factor is at once seen by developing the irrational part of it. For we have by the binomial theorem

$$\sqrt{B^2 - 4AC} = B - \frac{2AC}{B} - \frac{2A^2C^2}{B^3} - \&c.$$

so that the entire numerator is

$$-2A \frac{C}{B} \left\{ 1 - \frac{AC}{B^2} - \&c. \right\}$$

This divided by the denominator $2A$ gives a quotient which, when $A = 0$, becomes $-\frac{C}{B}$, as above.

the quantity under the radical will be

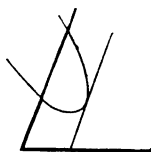
$$2(BD - 2AE)(x - \beta),$$

in which the factor $2(BD - 2AE)$ may be either positive, negative, or nothing.

If this factor be positive, the whole expression will be positive for every value of x greater than β , but negative for every value less than β ; hence, in this case, the locus extends indefinitely to the *right* of a parallel to the axis of y , drawn through the abscissa $x = \beta$; therefore this parallel is a tangent to the curve, to the left of which no point in the locus can be situated.



If the factor $2(BD - 2AE)$ be negative, then, on the contrary, the locus would extend indefinitely to the *left* of the parallel, whose abscissa is β ; and no point in the curve could be situated to the right of it.



In each of these cases, therefore, the curve will be limited in one direction, but unlimited in the opposite direction; it must therefore be a parabola.

If, lastly, $2(BD - 2AE) = 0$, then the expression for y becomes

$$y = -\frac{Bx + D}{2A} \pm \frac{1}{2A} \sqrt{D^2 - 4AF},$$

denoting a system of parallel straight lines, which, however, coincide, when $D^2 - 4AF = 0$, and which become imaginary when $D^2 - 4AF < 0$.

Because the condition $B^2 = 4AC$ or $B = 2\sqrt{AC}$ characterizes the parabola and its varieties, the first three terms in the general equation of this curve will always form a perfect square, viz.

$$(y\sqrt{A} + x\sqrt{C})^2 = Ay^2 + 2\sqrt{AC} \cdot xy + Cx^2.$$

(175.) We might now proceed to enquire into the form of the general equation when it represents one of the varieties of the central curves, and thence derive, as in the preceding chapter, (163,) criteria by means of which these varieties may be distinguished. For the varieties of the parabola the tests of their existence which we have just given, and which are the same as those established in the preceding chapter, (163,) are the simplest that can be employed, and may be readily applied in any case of doubt. But for the other curves, the shortest and most direct way of proceeding will generally be to solve the equation, with regard to one of the variables, and then to find the roots of that part of the resulting expression which is under the radical; the nature of these roots will make known the nature of the locus, conformably to the preceding discussion. The examples we shall here give will further illustrate this.

Construction of Curves of the second order.

EXAMPLE I.

To determine the position of the curve of which the equation is

$$y^2 - 2xy + 3x^2 + 2y - 4x - 3 = 0.$$

As, in this example, $B^2 - 4AC < 0$, the curve must be an ellipse; let us therefore first proceed to determine its limits. For this purpose let us put the equation under the following form, viz.

$$y^2 - 2(x-1)y = -3x^2 + 4x + 3 \dots (1),$$

which, solved first for y and then for x , gives

$$y = x - 1 \pm \sqrt{-2x^2 + 2x + 4} \dots (2),$$

$$x = \frac{y+2}{3} \pm \frac{1}{3} \sqrt{-2y^2 - 2y + 13} \dots (3)$$

Equating the irrational part of (2) to 0, we have

$$x^2 - x - 2 = 0 \therefore x = \frac{1 \pm 3}{2}$$

consequently, the roots of this equation being real and unequal, viz. $\beta=2$, and $\beta'=-1$, we know (170) that the curve exists; and that it is included between two parallels, LL' , MM' , to the axis of y , of which the abscissa, AG , of the one, is equal to -1 , and the abscissa, AH , of the other, equal to 2.

Solving, in like manner, the equation

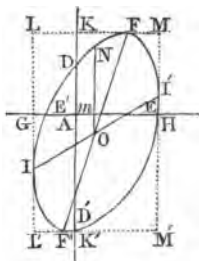
$$2y^2 + 2y - 13 = 0,$$

we obtain for the roots the values

$\frac{-1 + \sqrt{27}}{2}$, and $\frac{-1 - \sqrt{27}}{2}$; therefore the curve is also comprehended between two parallels to the axis of x , of which the ordinate, AK , of the one is $\frac{-1 + \sqrt{27}}{2}$, and the ordinate, AK' ,

of the other $\frac{-1 - \sqrt{27}}{2}$. The curve is therefore circumscribed by the parallelogram LM' .

To find the points of contact of the parallels LL' , MM' , we must construct the diameter, $Y = X - 1$ (168); for, as the abscissas β and β' of these points render the irrational part of the equation (2) nothing, the corresponding ordinates must belong as well to the diameter as to the curve. This diameter cuts the axes in the points $X = 1$ and $Y = -1$; if, therefore, through these points the line II' be drawn, the two points of contact will be determined. Constructing also the second diameter, $X = \frac{Y + 2}{3}$, which cuts the axes in the



points $x = \frac{2}{3}$ and $y = -2$, we obtain the other two points of contact, F, F'.

To find the points where the curve intersects the axis of y , suppose $x = 0$, in equation (2), and we have for the ordinates of those points $y = 1$ and $y = -3$; hence these points, D, D', are readily determined. In like manner, supposing $y = 0$, in equation (3), we have for the abscissas of the points E, E', where the curve cuts the axis of x , the values $x = \frac{2}{3} + \frac{1}{3}\sqrt{13}$ and $x = \frac{2}{3} - \frac{1}{3}\sqrt{13}$. The eight points thus determined are amply sufficient to make known the position of the curve.

But there is another mode of proceeding by which an indefinite number of points in the curve may be determined. Thus:

Draw, as before, the parallels LL', MM', and then construct the diameter, II', from its equation, $Y = X - 1$. The middle point, O, of this diameter is the centre of the curve, therefore the abscissa,

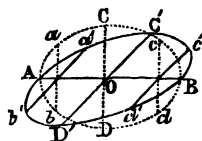
Am, of the centre is $\frac{\beta + \beta'}{2} = \frac{2 - 1}{2} = \frac{1}{2}$, because β and β' are the

abscissas of the *extremities* of the same diameter. Hence, drawing the ordinate mN, we shall have the direction of the diameter conjugate to II', since this ordinate will be parallel to the tangent at the vertex of that diameter; therefore, putting for x the value $x = \frac{1}{2}$, in the expression (2), the irrational part gives for the semi-diameter, ON,

$$\sqrt{-2x^2 + 2x + 4} = \frac{3\sqrt{2}}{2} = ON;$$

hence we have a system of conjugate diameters given in length and direction to construct the ellipse. This construction is as follows:

On the given diameters, AB, CD, taken as principal axes, construct an ellipse; then, if the double ordinates, ab , CD , cd , &c. of this ellipse be inclined to AB, in the given angle, while their length remains unchanged, their extremities, a' , b' , C' , D' , c' , d' , &c. will be all upon the required



curve, which may therefore be drawn through them. The truth of this is obvious, for the curve thus traced will, by construction, be such, that the squares of the chords parallel to one diameter, $C'D'$, are as the rectangles of the parts into which they divide the other, AB ; and AB , $C'D'$, are the given conjugates, both as to length and direction.

EXAMPLE II.

To construct the curve of which the equation is

$$y^2 - 2xy - 3x^2 - 2y + 7x - 1 = 0.$$

Since here $B^2 - 4AC > 0$, the curve is an hyperbola.

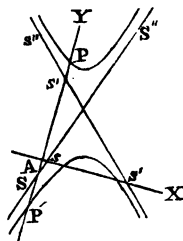
We shall proceed first to determine the asymptotes, because, when these are known, and a single point in the curve found, we can easily obtain as many more points in the curve as we please (104). The equation of the asymptotes is given at (172); but as it is advisable to proceed independently of the general formulas, we shall here deduce the equation of the asymptotes from the given equation of the curve, which furnishes for y the value

$$y = x + 1 \pm \sqrt{4x^2 - 5x + 2} = x + 1 \pm (2x - \frac{1}{4} + \frac{K}{x} + \frac{K'}{x^2} + \&c.)$$

Hence, for the two asymptotes we have the equation

$$Y = x + 1 \pm (2x - \frac{1}{4}).$$

For $x=0$ we have $Y = -\frac{1}{4}$, and $Y = 2\frac{1}{4}$; therefore, making $AS = -\frac{1}{4}$, and $AS' = 2\frac{1}{4}$, the points S , S' will be those in which the asymptotes cut the axis of y . In like manner, for $Y=0$ we have $x = \frac{1}{13}$, and $x = 2\frac{1}{4}$; therefore, making $As = \frac{1}{13}$, and $As' = 2\frac{1}{4}$, the points s , s' will be those in which the asymptotes cut the axis of x ; consequently the



lines SS'' , $s's''$ are the asymptotes sought. It remains now to determine a point in the curve, and for this purpose suppose $x=0$, in the proposed equation, and there results for the ordinates of the points where the curve intersects the axis of y , the values $y = 1 \pm \sqrt{2}$; therefore, making $AP = 1 + \sqrt{2}$, and $AP' = 1 - \sqrt{2}$, two points, P, P' , in the curve will be determined and thence as many more as we please (104).

When the axes of reference do not meet the curve, a point must be determined, by constructing the value of y , corresponding to an assumed value of x .

EXAMPLE III.

To construct the curve of which the equation is

$$xy - 2y + x - 1 = 0.$$

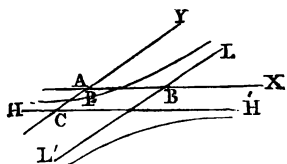
This equation represents an hyperbola, the axes of coordinates being parallel to the asymptotes (172). The expression for y is

$$y = -\frac{x-1}{x-2}$$

which becomes infinite only when $x = 2$; therefore, if AB be made equal to 2, the line LBL' , parallel to the axis of y , will be one of the asymptotes. In like manner, the expression for x , viz.

$$x = \frac{2y+1}{y+1}$$

becomes infinite only when $y = -1$; hence, if $AC = -1$, the line HCH' , parallel to the axis of x , will be the other asymptote.



To find a point in the curve, suppose $x = 0$, in the proposed equation, then $y = -\frac{1}{2}$; therefore, making $AP = -\frac{1}{2}$, the point **P** will be in the curve, and the construction will be effected as before.

EXAMPLE IV.

To construct the locus of the equation

$$y^2 - 4xy + 4x^2 + 2y - 7x - 1 = 0.$$

This curve is a parabola, because $B^2 - 4AC = 0$.

By solving the equation, first for y , and then for x , we have

$$y = 2x - 1 \pm \sqrt{3x + 2} \dots (1)$$

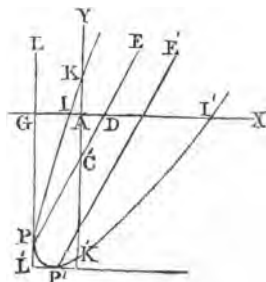
$$x = \frac{1}{2}y + \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{1}{3}y + \frac{4}{3}} \dots (2).$$

Equating the irrational part of (1) to 0, we have

$$3x + 2 = 0 \therefore x = -\frac{2}{3};$$

consequently the line, LL' , of which the abscissa, AG , is $-\frac{2}{3}$, will be a tangent to the curve, which will be situated to the *right* of this tangent, because the coefficient 3 of x under the radical is *positive*.

To find the point of contact we must construct the diameter $Y = 2X - 1$; for, as the abscissa, β , of this point renders the irrational part of (1) nothing, the corresponding ordinate must belong as well to this diameter as to the curve. Hence, supposing first $X = 0$, and then $Y = 0$, in the equation of the diameter, we have for the points where it cuts the

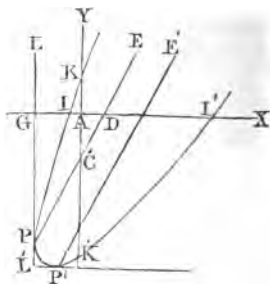


axes, $Y = -1$, and $X = \frac{1}{2}$; therefore, making $AC = -1$, and $AD = \frac{1}{2}$, and then, drawing the diameter, EP , we shall have the point of contact, P .

Equating, in like manner, the irrational part of the expression (2) to 0, we have

$$\frac{3}{4}y + \frac{1}{16} = 0 \therefore y = -2\frac{1}{16};$$

hence a parallel to the axis of x , drawn through the point P' , of which the ordinate is $-2\frac{1}{16}$, will be also a tangent to the curve. The point of contact will be determined by constructing the diameter, $E'P'$, from its equation, $X = \frac{1}{4}Y + \frac{1}{4}$.



To determine the points where the curve intersects the axis of x , suppose $y = 0$, in the proposed equation, and there results for the abscissas of those points the values $x = \frac{7 \pm \sqrt{65}}{8}$; hence these

points, I, I' , are readily determined. In like manner, putting $x = 0$, we have for the ordinates of the points K, K' , where the curve cuts the axis of y , the values $y = -1 \pm \sqrt{2}$. The points thus found are sufficient to determine the track of the curve; but others, if required, may be found by assuming different values for x , in (1), and constructing the resulting values for y .

(176.) We shall terminate this chapter with a table of the conditions which must exist among the coefficients of the general equation of the second degree, in order that the locus may meet the axes of reference. The necessity of the several conditions in the various cases is obvious, from an inspection of the general values of x and y , exhibited in art. (168).

When $E^2 - 4CF > 0$, the locus has two points of intersection with the axis of x .

When $E^2 - 4CF = 0$, the locus has one point of contact with the axis of x .

When $E^2 - 4CF < 0$, the locus has no point of intersection or of contact with the axis of x .

When $D^2 - 4AF > 0$, the locus has two points of intersection with the axis of y .

When $D^2 - 4AF = 0$, the locus has one point of contact with the axis of y .

When $D^2 - 4AF < 0$, the locus has no point of intersection or of contact with the axis of y .

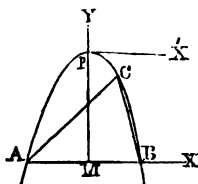
CHAPTER III.

PROBLEMS ON GEOMETRIC LOCI.

PROBLEM I.

(177.) Given the base of a triangle and the sum of the tangents of the angles at the base, to determine the locus of the vertex.

Let AB be the given base, through M , the middle of which, draw the perpendicular, MY ; then, taking MX , MY for axes, and denoting the vertex of the triangle by (x, y) , half the base by b , and the sum of the tangents by s , we have



$$\tan \angle A = \frac{y}{b+x}, \tan \angle B = \frac{y}{b-x}$$

$$\therefore s = \frac{2by}{b^2 - x^2}$$

consequently

$$sx^2 + 2by - sb^2 = 0, \text{ or } sx^2 + 2b(y - \frac{sb}{2}) = 0.$$

Hence the locus is a parabola.

By removing the origin to a point, P, in the axis of y , of which the ordinate is $\frac{sb}{2}$, that is, by substituting $y + \frac{sb}{2}$ for y , in the equation of the locus, it becomes

$$sx^2 + 2by = 0, \quad \text{or} \quad x^2 = -\frac{2b}{s}y;$$

so that P is the vertex of the curve, and PX' , PM , its principal axes. If we substitute $-\frac{sb}{2}$ for y , there result for x the values $x = \pm b$; hence the curve passes through the extremities of the base, A, B.

PROBLEM II.

(178.) Given the base and the difference of the tangents of the angles at the base, to determine the locus of the vertex.

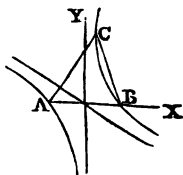
Taking the same axes as before, we have

$$\tan \angle B - \tan \angle A = \frac{2xy}{b^2 - x^2} = d,$$

d being put for the difference of the tangents.
Hence the equation of the locus is

$$2xy + dx^2 - db^2 = 0,$$

which belongs to an hyperbola; and, since the terms containing y^2 and y are absent from this equation, it follows (172) that the axis of y coincides with an asymptote; and since, moreover, the term containing x is also absent, the origin is at the centre. If $\pm b$ be substituted for x , in the equation of the locus, the resulting value of y is 0; hence the curve passes through the extremities of the base.



The other asymptote may be constructed by means of the expression at (173), which gives for the tangent of the angle α , which it makes with the axis of x , the value $\tan \alpha = -\frac{d}{2}$.

PROBLEM III.

(179.) Given the base of a triangle and the difference of the angles at the base, to determine the locus of the vertex.

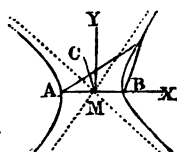
Taking the same axes as before, and putting a, a' , for the tangents of the angles at the base, and t for the tangent of their difference, we have

$$t = \frac{a - a'}{aa' + 1}$$

or substituting for a, a' , their respective values in terms of the coordinates of the vertex, as given in the first problem, the expression becomes

$$t = \frac{2xy}{y^2 - x^2 + b^2}$$

$$\therefore x^2 + \frac{2}{t}xy - y^2 - b^2 = 0.$$



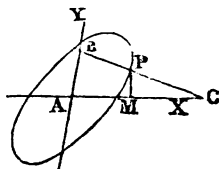
Consequently the locus is an hyperbola; and because the terms containing the first power of the variables are absent, the origin is at the centre. Also, since the coefficients of x^2 and y^2 are equal, and opposite in sign, the hyperbola is equilateral (173). It passes through the extremities of the base, since for $x = \pm b$, $y = 0$. When the vertex coincides with B, the angle A is 0, and the angle B is that contained by AB, and a tangent to the curve at B; this angle therefore is equal to the given difference; consequently, if MC make an angle with AM, equal to the difference of the angles at the base of the triangle, MC, being parallel

to the tangent at B, will be in the direction of the conjugate to AB; therefore the lines which bisect the angles CMA, CMB, will be the asymptotes to the curve (92).

PROBLEM IV.

(180.) It is required to find the locus of a given point in a straight line of given length, of which the extremities move along the sides of a given angle.

Let AX, AY, be the sides of the given angle, BC the given line, and P the given point; then, drawing the ordinate PM = y, and putting CP = a, PB = b, and the cosine of the angle A = c, we have (*Trig.* p. 19)



$$a^2 = MC^2 + y^2 - 2MC \cdot cy;$$

but

$$b : a :: x : MC = \frac{ax}{b}$$

hence, by substitution,

$$a^2 = \frac{a^2 x^2}{b^2} + y^2 - \frac{2acxy}{b}$$

$$\therefore a^2 x^2 - 2abctxy + b^2 y^2 - a^2 b^2 = 0.$$

Hence the locus is an ellipse, of which the centre is at the origin.

If the angle A is right, then $c = 0$, and the equation is

$$a^2 x^2 + b^2 y^2 = a^2 b^2;$$

in this case, therefore, the principal diameters of the curve coincide with the sides of the given angle. Hence is suggested an easy method of tracing an ellipse; thus, having drawn the perpendicular lines, AX, AY, apply to them the extremities of a rule, BC, of

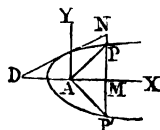
which the parts BP, PC, are respectively equal to the semi-minor and semi-major axes of the proposed curve; then in every such position of BC, P will mark a point in the curve. On this property of the ellipse is founded the *Trammel* or *Elliptic Compasses*.

PROBLEM V.

(181.) Two straight lines are given in position, from any point, in one of which, a perpendicular is drawn to the other, and from a given point in this latter, with a radius equal to the perpendicular, an arc, cutting the perpendicular in P, is described. It is required to find the locus of the point P.

Let DX, DN, be the lines given in position, NM a perpendicular from the latter to the former, and A the given point; then we must always have $AP = NM$.

Put $NM = Y$, $PM = y$, $AD = p$, and the tangent of the angle $D = a$; then, taking the rectangular axes, AX, AY, we have for the equation of DN passing through the point $(-p, 0)$.



$$Y = a(x + p) \therefore Y^2 = a^2x^2 + 2a^2px + a^2p^2;$$

but

$$Y^2 = x^2 + y^2,$$

$$\therefore x^2 + y^2 = a^2x^2 + 2a^2px + a^2p^2;$$

hence the equation of the locus is

$$y^2 + (1 - a^2)x^2 - 2a^2px - a^2p^2 = 0 \dots (1).$$

If the angle D is 45° , then $a = 1$, and the equation is

$$y^2 = 2px + p^2 = 2p\left(x + \frac{p}{2}\right),$$

which characterizes a parabola, of which the abscissa of the vertex is $-\frac{p}{2}$; that is, the vertex is at the middle of AD, and since p is also the semi-parameter, it follows that A is the focus.

If the angle D is less than 45° , then $a < 1$, and the locus is an ellipse, and because y enters in the equation only in its second power, there are two equal values of y for one value of x ; hence the axis of x is a principal diameter of the curve. If the equation be solved for x , the rational part of the resulting expression will be

$$\frac{a^2 p}{1-a^2}$$

this, therefore, (168) is the value of the abscissa of the centre; by substituting it for x , in the equation of the locus, the resulting value of y , viz.

$$\frac{ap}{\sqrt{1-a^2}}$$

will be the length of the semi-diameter, parallel to the axis of y . For the length of the other principal semi-diameter, take half the difference of the two values of x , which the equation gives for $y = 0$, and we obtain the expression

$$\frac{ap}{1-a^2}.$$

This semi-diameter may however be found rather more easily; for since, in the equation of the locus, when the origin of the axes is removed to the centre, x^2 will preserve the same coefficient, it follows that, denoting the transformed by

$$y^2 + \frac{B^2}{A^2} x^2 = A^2 B^2 \dots (2),$$

the coefficients of x^2 in (1) and (2) will be identical; that is,

we must have $\frac{B^2}{A^2} = 1 - a^2$, but we have found $B^2 = \frac{a^2 p^2}{1-a^2}$; hence

$$A^2 = \frac{a^2 p^2}{(1-a^2)^2} \therefore A = \frac{ap}{1-a^2}.$$

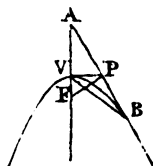
If the angle D is greater than 45° , the locus of P is an hyperbola, of which the centre and axes may be determined as in the case of the ellipse.

When the given lines are parallel, the locus is obviously a circle, because then MN or AP is constant.

PROBLEM VI.

(182.) To find the locus of the vertex of a parabola which shall touch a given straight line, and have a given focus.

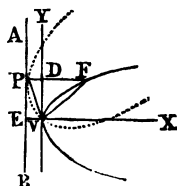
Let AB be the given straight line, and F , the focus; draw the perpendicular, FP , and through the vertex, V , draw FVA , join also PV . Then (116) $FA \cdot FV = FP^2$, therefore PVF must be a right angle; consequently the locus of V is a circle, of which the diameter is FP



PROBLEM VII.

(183.) To find the locus of the focus of a parabola which shall touch a given straight line, and have a given vertex.

Let V be the given vertex, and AB the given tangent; then, for every position, F , of the focus, the perpendicular, FP , subtends a right angle at the vertex, V . Let VY , parallel to AB , be taken for the axis of y , and VX , perpendicular to it, for the axis of x ; then, by similar triangles, VDF , PDV ,



we have

$$VD = y : DF = x :: PD = a : y$$

$$\therefore y^2 = ax.$$

Hence the locus is a parabola, of which the axes are VX, VY, and parameter PD or VE.

PROBLEM VIII.

(184.) Given the base and altitude of a triangle to find the locus of the intersection of perpendiculars from the angles to the opposite sides.

Let c represent the base, AB , of the triangle; then the altitude, a , being constant, the locus of the vertex, C , is a parallel to AB . Hence, taking AB , AY , for the rectangular axes, and putting (x', y') for any point in the locus of C , we have always $y' = a$; and for the equation of BC , passing through the points $(c, 0)$ and (x', y') , we have

$$y = \frac{y'}{x' - c} (x - c),$$

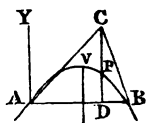
also for the equation of a perpendicular to this, through the origin, we have

$$y = \frac{c - x'}{y'} x.$$

At the point P , where this line intersects the perpendicular, CD , $x = x'$; therefore, substituting x for x' , and a for y' , in the foregoing equation, we have for the locus of P the equation

$$ay = cx - x^2,$$

which characterizes a parabola.



For $x = 0$ we have $y = 0$; therefore the curve passes through A, but does not again meet AY; so that AY is a diameter. For $y = 0$ we have not only $x = 0$, but also $x = c$; therefore, the curve passes through B; hence the principal diameter bisects AB at right angles; therefore, to find the vertex, put $x = \frac{1}{2}c$, which gives $y = \frac{c^2}{4a}$.

By removing the origin of the axes (40) to the vertex, that is, to the point $(\frac{c}{2}, \frac{c^2}{4a})$, the equation becomes

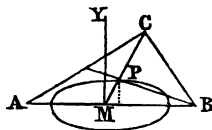
$$ay = -x^2;$$

therefore the parameter is $a = CD$.

PROBLEM IX.

(185.) Given the base and the sum of the sides of a triangle, to find the locus of the point of intersection of lines from the angles bisecting the opposite sides.

Let M be the middle of the base, AB, and take MB, MY, for rectangular axes. Put (x', y') for the vertex, C, of the triangle, and (x, y) for P, one of the points in the locus, then (20)



$$y = \frac{y'}{3} \therefore x = \frac{x'}{3} \therefore (x', y') = (3x, 3y).$$

Now the locus of (x', y') is an ellipse, of which the principal diameter, $2A$, is equal to the sum of the given sides of the triangle, and the foci A and B. The equation of the locus of (x', y') is therefore

$$A^2 y'^2 + B^2 x'^2 = A^2 B^2;$$

hence, by substitution, we have for the locus of (x, y) the equation

$$A^2 y^2 + B^2 x^2 = \frac{A^2 B^2}{9}$$

or

$$\left(\frac{A}{3}\right)^2 y^2 + \left(\frac{B}{3}\right)^2 x^2 = \left(\frac{A}{3}\right)^2 \cdot \left(\frac{B}{3}\right)^2,$$

an ellipse, of which the principal diameters are one-third those of the former.

If, instead of the sum, the difference of the sides had been given, the locus would have been an hyperbola; since, in that case, the locus of (x', y') would have been an hyperbola.

But, if the sum of the tangents of the angles at the base had been constant, then the locus would have been a parabola, (*Prob. 1.*)

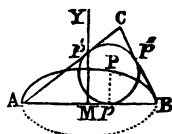
PROBLEM X.

(186.) Given the base and sum of the sides of a triangle, to find the locus of the centre of the inscribed circle.

Let AB be the given base, and P the centre of one of the circles, of which p is the point of contact with the base; then it is known that the distance of p from M, the middle of the base, is always equal to half the difference of the sides. This is easily proved; for, since two tangents drawn to a circle from any point are equal, we have $AC - CB = Ap' - Bp'' = Ap - Bp = 2Mp$. This being premised, take MB, MY, for rectangular axes, put $MB = c$, $C = (x', y')$, and $P = (x, y)$; then the area of the triangle, $ABC = y'c$, or putting A for half the given sum, $AC + BC$, the area of the same triangle is $y(A + c)$; consequently

$$y'c = y(A + c) \therefore y' = \frac{y(A + c)}{c}$$

Now, since the locus of C is an ellipse, of which A, B, are the foci,



and $2A$ the major diameter, we have (49)

$$\frac{1}{2}(AC - CB) = ex' = x \therefore x' = \frac{x}{e} = \frac{Ax}{c}.$$

Hence, substituting these values of x' and y' , in the equation of the locus of (x', y') , viz. in

$$A^2y'^2 + B^2x'^2 = A^2B^2,$$

we have, for the locus of P , the equation

$$(A + c)^2y^2 + B^2x^2 = B^2c^2,$$

which characterizes an ellipse, of which the axes coincide with the former. For $x = 0$ we have $y = \frac{Bc}{A + c}$, and for $y = 0$ we have $x = c$; these values of x and y are those of the principal semi-diameters of the locus.

If, instead of the sum, the difference of the sides had been given, then, since half this difference, that is, x , would have been constant, the locus of P would have been a straight line through p , perpendicular to the base.

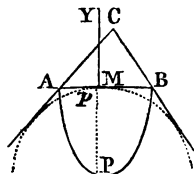
PROBLEM XI.

(187.) Given the base and the sum of the sides of a triangle to find the locus of the centre of the circle touching the base, and the prolongation of the other two sides.

Taking the same axes as in the last problem, let P be the centre of one of the circles, and p its point of contact with the base; then, as before,

$$Mp = -x = \frac{1}{2}(AC - CB) = ex' \therefore x' = -\frac{Ax}{c}$$

Now, as the centre of the circle must always be on the line bisecting the angle C , that is, on the normal,



through the point (x', y') , we have, by substituting this value of x' , in the equation of the normal, the equation

$$y = \frac{B^2 y' - A c y' - A^2 y'}{B^2} = - \frac{(A + c) c y'}{A^2 - c^2} = \frac{-c}{A - c} y'$$

to determine y' ; which is

$$y' = - \frac{A - c}{c} y.$$

These values of x' and y' , substituted in the locus of (x', y') , give, for the locus sought, the equation

$$(A - c)^2 y^2 + B^2 x^2 = B^2 c^2,$$

which is that of an ellipse, of which the minor diameter is $2c = AB$,

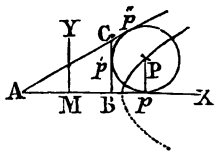
and major diameter $2 \frac{Bc}{A - c} = 2 \frac{\sqrt{A^2 - c^2} \cdot c}{A - c} = 2 \sqrt{\frac{A + c}{A - c}} \cdot c.$

If, instead of the sum, the difference of the sides had been given, then, since x would have been constant, the locus of P would have been a straight line through p , perpendicular to the base.

PROBLEM XII.

(188.) Given the base and the difference of the sides of a triangle, to find the locus of the centre of the circle touching one side, and the prolongation of the base and of the other side.

Let P be the centre of one of the circles, and p, p', p'' the several points of contact; then $Ap'' = Ap = AC + Cp' = AB + Bp'$
 $\therefore Ap = \frac{1}{2}(AB + AC + BC)$, and, taking AM or $\frac{1}{2}AB$ from each side, we have $Mp = \frac{1}{2}(AC + BC) = x$:



Now, since the locus of C is an hyperbola, of which AC, BC, are radii vectores, we have (86)

$$\frac{1}{2}(AC + BC) = ex' = x \therefore x' = \frac{Ax}{c}.$$

Also, since P must always be on the line bisecting the angle BCP', that is to say, on the normal through the point (x', y'), we shall have to determine the value of y' from the equation of the normal,

when x' is replaced by $\frac{Ax}{c}$. This equation is

$$y = \frac{B^2y' - Axy' + A^2y'}{B^2} = \frac{(A - c)c}{A^2 - c^2}y' = \frac{c}{A + c}y',$$

$$\therefore y' = \frac{A + c}{c}y.$$

These values of x' and y', substituted in the locus of (x', y'), give for the locus sought the equation

$$(A + c)^2y^2 - B^2x = -B^2c^2,$$

which characterizes an hyperbola, whose principal axes are

$$2c \text{ and } \frac{2Bc}{A + c} \sqrt{-1}.$$

If, instead of the difference, the sum of the sides had been given, then, since half this sum, or x, is constant, the locus would have been a straight line perpendicular to, and through the extremity of, the major diameter of the ellipse, which is the locus of C.

PROBLEM XIII.

(189.) Two straight lines are perpendicular to each other, and through two given points in one, straight lines are drawn, forming,

with the other, angles, the product of whose tangents is constant: what is the locus of their intersection?

Let the perpendicular lines be taken for axes, and let the equations of any pair of the intersecting lines be

$$y = ax + b$$

$$y = ax + \beta$$

Then, by the conditions of the problem, the quantities ax , b , and β , are constant; hence, multiplying the two equations together, and reducing, we have, for the equation of the locus,

$$y^2 - aax^2 - (b + \beta)y + b\beta = 0,$$

which characterizes an hyperbola, if ax is positive, and an ellipse, if ax is negative.

Because x enters into this equation only in its second power, there are for every value of y two equal values of x ; therefore the axis of y is a principal diameter of the curve. If we put $x = 0$, the resulting values of y are obviously b and β ; the difference of these is the value of the diameter, which coincides with the axis of y , that is

$$A = \frac{b - \beta}{2}.$$

Also the sum of the same values gives twice the ordinate of the centre, therefore

$$Y = \frac{b + \beta}{2}.$$

To find the other principal diameter, put this value of Y for y , in the equation of the locus, and there results

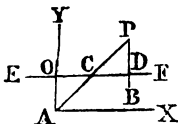
$$x = B = \frac{b - \beta}{2\sqrt{-aa}}.$$

If $ax = +1$, the locus is an equilateral hyperbola, and, if $ax = -1$, it is a circle. In every case, the part of the axis intercepted by the given points is a principal diameter of the curve, as the foregoing value of A proves.

PROBLEM XIV.

(190.) From two given points two straight lines are drawn so as to intercept a given portion of a straight line given in position: what is the locus of the intersection of these lines?

Let EF be the line given in position, and A, B , the given points; let also AP, BP , be two lines intercepting the given portion, $CD = m$, then P is a point in the locus. Draw the axes AX, AY , the one parallel and the other perpendicular to EF , put $AO = p$ and (x', y') for the point B , then the equation of AP is



$$y = ax \therefore \text{when } y = p, OC = \frac{p}{a}$$

also the equation of BP is

$$y - y' = a'(x - x'), \therefore \text{when } y = p, OD = \frac{p - y' + ax'}{a'}$$

$$\therefore OD - OC = CD = \frac{p - y' + ax'}{a'} - \frac{p}{a} = m,$$

that is, substituting $\frac{y}{x}$ for a , and $\frac{y - y'}{x - x'}$, for a' ,

$$\frac{(p - y')(x - x')}{y - y'} + x' - \frac{px}{y} = m,$$

this equation becomes, after reduction,

$$(x' - m)y^2 - y'xy + (my' - px')y + py'x = 0.$$

Hence the locus is an hyperbola.

As, in this equation, the square of one of the variables, viz. x^2 , is absent, the axis of x is parallel to an asymptote (172), the ordinate of which is $\frac{py'}{y} = p$; hence the line EF is that asymptote.

To determine the centre, we may solve the equation of the locus with regard to y , and, by omitting the irrational part, in the resulting expression for y , we shall have the equation of a diameter, in which, by putting p for y , we shall obtain for x the abscissa of the centre, which is therefore thus determined. Having found the centre, we may construct the other asymptote; thus, assume any value for x , and construct the two values of y corresponding; two points of the curve will be thus determined, either of which is at the same distance from the known asymptote that the other is from the asymptote sought, the distances being measured along the line passing through the two points, and in opposite directions; hence the centre, and a point in the asymptote, being found, the line may be drawn. Or, without first finding the centre, we may determine in this way two points in the required asymptote, which will determine its position.

PROBLEM XV.

(191.) Tangents to a parabola form a given angle with each other: what is the locus of their point of intersection?

Let t represent the tangent of the given angle, and (x, y) any point of intersection; then, if the equation of the parabola be $y^2 = 2mx$, the equations of tangents through any points (x', y') , (x'', y'') of the curve will be (106)

$$yy' = m(x + x'), \quad yy'' = m(x + x'').$$

As the trigonometrical tangents of the angles which these tangents form with the axis of x are respectively $\frac{m}{y'}$, $\frac{m}{y''}$, we have for t , the tangent of their difference, the expression

$$t = \frac{m(y'' - y')}{y'y'' + m^2};$$

it remains therefore to determine y' , y'' , in terms of x , y . For this purpose, substitute for $2mx'$, $2mx''$, in the equations of the tangents, their equals y'^2 , y''^2 , and we have the two equations

$$y'^2 - 2yy' + 2mx = 0, \quad y''^2 - 2yy'' + 2mx = 0,$$

in which the roots of the one are the same as those of the other; therefore, by the theory of equations,

$$y'y'' = 2mx \text{ and } y' + y'' = 2y \therefore (2y)^2 - 8mx = (y'' - y')^2;$$

hence, substituting in the square of the expression for t , we have

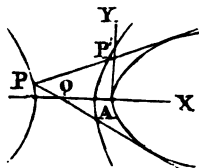
$$t^2 = \frac{m^2(4y^2 - 8mx)}{(2mx + m^2)^2} = \frac{4y^2 - 8mx}{(2x + m)^2}$$

$$\therefore y^2 - t^2x^2 - (2 + t^2)mx - \frac{1}{4}t^2m^2 = 0;$$

consequently the locus is an hyperbola, of which a principal diameter coincides with the axis of x , since there are two equal values of y for $x = 0$. If we put $y = 0$, in the equation, the difference of the roots will be the length of this diameter, and half their sum, the abscissa of the centre; this abscissa, therefore, is

$$- \left(\frac{1}{t^2} + \frac{1}{2} \right) m = - (\cot^2 P + \frac{1}{2}) m = AO.$$

By substituting it for x , in the equation of the locus, we get for



the square of the semidiameter parallel to the axis of y the expression

$$y^2 = -m^2 \left(\frac{1}{t^2} + 1 \right) = -m^2 (\cot^2 P + 1) = -m^2 \operatorname{cosec}^2 P.$$

Now, instead of determining the other diameter by taking the difference of the roots of the equation, as above suggested, we shall obtain it more readily from these considerations. We know that if we had removed the origin of the axes to the centre of the curve, that is, if, in the equation of the locus, we had substituted $x - (\cot^2 P + \frac{1}{2})m$ for x , the equation would have been transformed to the form

$$y^2 - \frac{B^2}{A^2} x^2 = -B^2.$$

But, in this transformed equation, the coefficient of x^2 will be the same as in the primitive, viz. $-t^2$; consequently,

$$\frac{B^2}{A^2} = \tan^2 P, \text{ and } B^2 = m^2 \operatorname{cosec}^2 P,$$

$$\therefore A^2 = \frac{m^2 \operatorname{cosec}^2 P}{\tan^2 P}, \therefore A = \frac{m \operatorname{cosec} P}{\tan P}$$

If $t = 1$, that is, if the given angle be 45° , the locus will be an equilateral hyperbola.

If $t = \infty$, that is, if the given angle be 90° , then the denominator, in the expression for t^2 , must be 0; that is, $2x + m = 0$,

or $x = -\frac{m}{2}$; in this case, therefore, the locus is the directrix of

the proposed parabola.

If t is negative, that is, if the given angle is obtuse, the equation of the locus will remain unaltered, since t enters only in its second power. Hence we infer that, if any pair of tangents intersect at an angle, P , and any other pair intersect at an angle, P' , supplementary to the former, the locus of P will be one branch of the hyperbola, and the locus of P' the other branch.

PROBLEM XVI.

(192.) Tangents to a parabola form angles with the principal diameter, the product of whose tangents is given: what is the locus of the points of intersection?

Let one of the points of intersection be (x, y) , and one of the points of contact (x', y') ; then, from the equation of the curve

$$y'^2 = 2mx' \therefore x' = \frac{y'^2}{2m} \} \dots (1),$$

and from the equation of the tangent,

$$yy' = mx + mx' = mx + \frac{y'^2}{2} \} \dots (2).$$

Also, for the value of a , the tangent of the angle which the tangent through (x', y') makes with the principal diameter, we have

$$a = \frac{m}{y'} \therefore y' = \frac{m}{a}.$$

Substituting this value of y' , in equation (2), and reducing, we have

$$a^2 - \frac{y}{x} a + \frac{m}{2x} = 0.$$

The two values of a , contained in this equation, belong to the two tangents drawn from the point (x, y) , and, as their product, p , is given, we have, by the theory of equations,

$$p = \frac{m}{2x} \therefore 2px = m, \text{ or } x = \frac{m}{2p}.$$

Hence the locus sought is a straight line, perpendicular to the principal diameter, and at the distance $\frac{m}{2p}$ from the vertex.

PROBLEM XVII.

(193.) To find the locus of the intersections of pairs of tangents to any line of the second order, when they make angles with the principal diameter, such that the product of their tangents may be a given quantity.

This problem has just been solved for the parabola, and may, by employing a similar process, be extended to the other two curves. Thus, representing a point of intersection by (x, y) , and a point of contact by (x', y') , we should have, from the equation of the curve,

$$A^2y'^2 \pm B^2x'^2 = \pm A^2B^2 \dots (1),$$

and, from the equation of the tangent,

$$A^2yy' \pm B^2xx' = \pm A^2B^2 \dots (2).$$

Moreover, the expression for a , the trigonometrical tangent of the angle, this line makes with the principal diameter is

$$a = \mp \frac{B^2 x'}{A^2 y'} \dots (3).$$

By eliminating x', y' from these equations, as in Problem v. p. 135 of Part I., we shall obtain the quadratic equation in a , there deduced; and as the product p of the tangents is given, we have

$$p = aa' = \frac{y^2 \mp B^2}{x^2 - A^2}$$

$$\therefore y^2 - px^2 = \pm B^2 - pA^2$$

which, according as p is negative or positive, is the equation of an ellipse or hyperbola, concentric with the proposed.

If, however, the proposed curve be an hyperbola, and p be

negative, and numerically less than $\frac{B^2}{A^2}$, then the equation of the locus is of the form $y^2 + px^2 = -P$, which is the representative of an imaginary curve (151); so that, under these conditions, no pair of tangents to the hyperbola can exist. If p be negative, and numerically equal to $\frac{B^2}{A^2}$, then, when the original curve is an hyperbola, the sought locus is expressed by the equation $y^2 + px^2 = 0$, which merely represents a point, the origin of the axes; that is, the centre of the proposed hyperbola. The tangents from this point are no other than the asymptotes which we already know (78) are inclined to the principal diameter at angles whose tangents a, a' have the property $aa' = -\frac{B^2}{A^2}$. If p , instead of being

$-\frac{B^2}{A^2}$, be $+\frac{B^2}{A^2}$, the locus will be an hyperbola, whose equation is $A^2y^2 - B^2x^2 = -2B^2$, and the absolute lengths of whose semi-axes are, therefore, $A\sqrt{2}$ and $B\sqrt{2}$. When the original curve is an ellipse, then the locus is an ellipse, with these same semi-axes.

But in order to give a general investigation of this problem, or one that shall comprehend all the three curves, we shall proceed in the following manner.

The general equation of a line of the second order, when referred to the principal diameter, and tangent through its vertex, is $y^2 = mx + nx^2$; therefore, from the equation of the curve, we have

$$y'^2 = mx' + nx'^2 \dots (1),$$

and for the tangent, a , of the angle, formed by the axis of x , and a straight line through the points (x, y) and (x', y') , we have the expression

$$a = \frac{y - y'}{x - x'} \therefore y' = y - ax + ax' \dots (2).$$

Substituting this value of y' in (1), and arranging the result according to the powers of x' , we have the equation

$$(a^2 - n)x'^2 + (2ay - 2a^2x - m)x' + y^2 - 2axy + a^2x^2 = 0 \dots (3)$$

This equation gives two values for x' ; but since, by the conditions of the problem, the line through the points (x, y) , (x', y') must have only one point, viz. (x', y') , in common with the curve, the two values of x' , in (3), must be equal; in other words, the first member of the equation must be a complete square. Hence, by the known composition of a square,

$$4(a^2 - n)(y^2 - 2axy + a^2x^2) = (2ay - 2a^2x - m)^2.$$

Reducing this equation, and arranging the result according to the powers of a , we obtain finally

$$a^2 - \frac{my + 2nxy}{mx + nx^2}a + \frac{m^2 + 4ny^2}{4(mx + nx^2)} = 0.$$

The two values of a , contained in this equation, belong to the two tangents drawn from the point (x, y) ; and, since their product p is given, we have, by the theory of equations,

$$p = \frac{m^2 + 4ny^2}{4(mx + nx^2)}$$

hence

$$ny^2 - pnx^2 - pmx + \frac{m^2}{4} = 0 \dots (4),$$

the equation of the locus required, which is, therefore, an hyperbola, or an ellipse, according as p is positive or negative. When, however, $n = 0$, that is, when the proposed curve is a parabola, this locus becomes a straight line, in which

$$x = \frac{m}{4p}$$

shewing that it is perpendicular to the axis, and that it coincides

with the directrix when $p = -1$, or when the intersecting tangents include a right angle. Since equation (4) gives two equal values of y for $x = 0$, it follows that a principal diameter of the locus coincides with the axis of x . If we put $y = 0$, in the equation, half the sum of the roots of the resulting equation in x will be the abscissa of the centre; this abscissa is therefore $-\frac{m}{2n}$, which is no other than the abscissa of the centre of the original curve, when this is either an ellipse or hyperbola; because, from the equations of these curves, putting a, b for their semi-diameters, we have

$$-m \div 2n = \frac{2b^2}{a} \div \frac{2b^2}{a^2} = a,$$

so that the curves are concentric.

Substituting the foregoing general value of x , the abscissa of the centre of the locus, in (4), we have for the square of the semi-diameter, B , parallel to the axis of y ,

$$y^2 = B^2 = -\frac{m^2(p+n)}{4n^2}$$

But if the axes of reference be removed to the centre of the locus, by substituting in (4) $x + a$ for x , the coefficient of x^2 will remain unaltered; hence $\frac{B^2}{A^2} = -\frac{pn}{n} = -p$, consequently, dividing the expression for B^2 by $-p$, we have

$$A^2 = +\frac{m^2(p+n)}{4n^2p}$$

therefore, if n be positive, and p negative, but numerically less than n , the locus will be impossible; for, in this case, the foregoing expressions for the squares of the semi-axes will both be negative; if, however, instead of being numerically less, p be numerically equal to n , then the semi-axes vanish, and the locus becomes a point.

If $p = -1$, that is, if the intersecting tangents to the ellipse or hyperbola form a right angle, the locus will be a circle, of which the radius is

$$A = B = \frac{\sqrt{m^2(1-n)}}{2n}$$

But if $p = +1$, the locus will be an equilateral hyperbola, of which the principal semi-transverse is given by the same expression.

If we suppose the intersecting tangents to be parallel to conjugate diameters, then $p = n$ (52, 89) $= \mp \frac{b^2}{a^2}$; still putting a and b for the principal semi-conjugates of the proposed ellipse, or hyperbola; also $m = \pm \frac{2b^{3*}}{a}$; therefore, by these substitutions, in equation (4), the locus becomes

$$y^2 \pm \frac{b^2}{a^2} x^2 \mp \frac{2b^3}{a} x \mp b^2 = 0,$$

which characterizes an ellipse, when the proposed curve is an ellipse; that is, when p is negative, or the upper sign has place; but when the original curve is an hyperbola, and, consequently, the lower signs only apply, then the locus is likewise an hyperbola. The semi-axes of the locus are, in both cases, $A \sqrt{2}$ and $B \sqrt{2}$, as we readily see by removing the origin to the centre of the curve, that is, by substituting $x + a$ for x .

PROBLEM XVIII.

(194.) What is the locus of the centres of all the circles which pass through a given point and touch a given straight line?

Ans. A parabola.

* It must be recollected that in the present problem the axes of reference are supposed to originate at the *left hand* vertex of the proposed curve; so that, in the equation of the hyperbola (81), A will be *minus*.

PROBLEM XIX.

What is the locus of the centres of all the circles which may touch two given circles? *Ans.* An hyperbola.

PROBLEM XX.

The directrix and a point in a parabola being given to determine the locus of the vertex. *Ans.* An ellipse.

PROBLEM XXI.

From the extremities of the major diameter of an ellipse, two straight lines are drawn terminating in the opposite extremities of any double ordinate to the diameter. These lines, when produced, meet in a point, of which the locus is required.

Ans. An hyperbola.

PROBLEM XXII.

Upon a given base triangles are constructed, having always one angle at the base double the other; what is the locus of their vertices? *Ans.* An hyperbola.

PROBLEM XXIII.

From any point in a given straight line two straight lines are drawn, the one perpendicular to the given line, and the other to a given point; if the perpendicular be made equal to the other, what will be the locus of its extremity?

Ans. An equilateral hyperbola.

PROBLEM XXIV.*

From the centre of a given semicircle lines are drawn to the circumference; each of these is divided in P, so that the distance of P from the circumference, measured on the line, is to its distance from the base of the semicircle in a constant ratio. Prove that when this latter distance is *equal* to the former, the locus of P is a parabola; when it is *greater*, the locus is an ellipse; and when *less*, an hyperbola; and that in each case the diameter of the semicircle is the principal parameter of the locus.

PROBLEM XXV.

To determine the curve of which each ordinate is a mean between the corresponding ordinates of two given straight lines.

CHAPTER IV.

MISCELLANEOUS PROPOSITIONS.

PROBLEM I.

(195). To determine the equation of a straight line passing through a given point, and making a given angle with a given straight line, the axes of coordinates being oblique.

* If the remaining semicircle be supplied, and the point be taken on the prolongation of its radii, still preserving the constant ratio between its distance from the circumference and from the diameter, the point will generate a continuation of the curve.

Let β be the inclination of the axes, (x', y') the given point, and v the tangent of the given angle. The equations of the given and required lines will be of the forms

$$y = ax + b$$

$$y - y' = a'(x - x')$$

respectively; and the tangent v of the angle at which they intersect is (p. 37, Part I.)

$$v = \frac{(a \sim a') \sin \beta}{1 + aa' + (a + a') \cos \beta}$$

from which we easily get

$$a' = \frac{a \sin \beta \mp v(1 + a \cos \beta)}{\sin \beta \pm v(a + \cos \beta)};$$

and hence the equation of the required line is

$$y - y' = \frac{a \sin \beta - v(1 + a \cos \beta)}{\sin \beta + v(a + \cos \beta)} (x - x')$$

or else

$$y - y' = \frac{a \sin \beta + v(1 + a \cos \beta)}{\sin \beta - v(a + \cos \beta)} (x - x')$$

according as the angle is formed to the left or to the right of the given line (see *art.* 14).

When the lines are perpendicular $v = \frac{1}{b}$, therefore

$$1 + aa' + (a + a') \cos \beta = 0$$

$$\therefore a' = -\frac{1 + a \cos \beta}{a + \cos \beta}$$

in which case the equation of the required line is

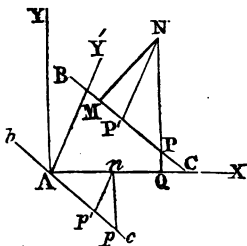
$$y - y' = -\frac{1 + a \cos \beta}{a + \cos \beta} (x - x')$$

PROBLEM II.

(196.) To find the expression for the length of the perpendicular from a given point to a given line; the axes of reference being oblique.

The required expression may be determined by proceeding as in (16), using the final equation of last problem for that of the perpendicular instead of the equation marked (3) in the article referred to; and then employing the general expression for D in (15), instead of that peculiar to rectangular axes. The process will, however, be encumbered with complicated expressions, so that we shall prefer to deduce the expression sought from the particular case established at (16); and this we may do with considerable facility.

Referring to the expression for MN or D at (16), we at once see that the numerator of that expression, viz. $y' - ax' - b$ is no other than the analytical value of NP , the parallel to the axis of y , drawn from the given point to the given line; for y' is the given ordinate NQ ; and $ax' + b$ is by (2) the ordinate PQ of the given line, to the same abscissa x' . This is evidently the interpretation of $y' - ax' - b$ whatever be the axes of reference. When the axes are oblique, as AX, AY' in the annexed figure, then this expression is the value of NP' . If, then, we can find an expression for NP in terms of NP' and given quantities, the numerator of the fraction for MN may be replaced by a given function of NP' . As to the denominator, we readily obtain an equivalent for it in terms of the new inclinations from the expression for $\tan \alpha$ at (13). This denominator is



$$\sqrt{1 + \tan^2 \alpha} = \sqrt{\left\{1 + \frac{a^2 \sin^2 \beta}{(1 + a \cos \beta)^2}\right\}} = \frac{\sqrt{1 + 2a \cos \beta + a^2}}{1 + a \cos \beta} \dots (1)$$

Now the ratio $\frac{NP}{NP'}$ is evidently the same as $\frac{np}{np'}$, given by two lines drawn from any point n , in the axis of x , parallel to the former two from N , and terminating in bc through the origin and parallel to BC . Moreover, by the rectangular coordinates, $np = \tan \alpha \cdot An$; by the oblique coordinates, $np' = a \cdot An$; hence

$$\frac{NP}{NP'} = \frac{\tan \alpha}{a} = \frac{\sin \beta}{1 + a \cos \beta}, \text{ by art. (13)}$$

$$\therefore NP \text{ or } y' - \tan \alpha \cdot x' - b = NP' \frac{\sin \beta}{1 + a \cos \beta} \dots (2).$$

Therefore, this being the numerator, and the expression (1) the denominator, in the fraction for MN at (16), it follows that, when the axes are oblique, that fraction is

$$MN = \frac{(y' - ax' - b) \sin \beta}{\sqrt{1 + 2a \cos \beta + a^2}}$$

the first factor in the numerator being the analytical value of NP' , as shewn above; the x' , y' and b belonging now to the oblique axes, and a being the coefficient of x in the equation of the given straight line.

THEOREM.

(197.) If to any line of the second order two secants, parallel to the sides of a given angle, be drawn, then the two rectangles contained by the parts intercepted between their point of intersection and the curve will have a constant ratio, wherever that point of intersection may be.

Let AX, AY be parallel to the sides of a given angle, A', and intersect any line of the second order, in the points P, P', and p, p' .

The equation of the curve referred to these lines as axes is

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0.$$

in which, if we put $y = 0$, the resulting values of x will be the abscissas AP, AP', that is, these lines will be given by the roots of the equation

$$Cx^2 + Ex + F = 0,$$

or

$$x^2 + \frac{E}{C}x + \frac{F}{C} = 0,$$

$$\therefore AP \cdot AP' = \frac{F}{C}.$$

In like manner, putting $x = 0$, in the equation of the curve, the resulting values of y will determine the ordinates Ap, Ap', that is, these lines will be determined by the roots of the equation

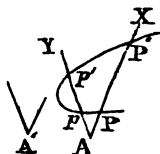
$$y^2 + \frac{D}{A}y + \frac{F}{A} = 0,$$

$$\therefore Ap \cdot Ap' = \frac{F}{A}.$$

consequently

$$AP \cdot AP' : Ap \cdot Ap' :: \frac{F}{C} : \frac{F}{A} :: A : C.$$

Now no change can take place in the coefficients A and C of y^2 and x^2 , by removing the origin, A, without altering the inclination of the axes; that is to say, these coefficients will remain the same, although we put $y + b$ for y , and $x + a$ for x ; consequently, so long as the two secants remain parallel to the two given lines, the ratio of the rectangles AP . AP', Ap . Ap' will be the same, wherever A may be.



In the central curves the constant ratio will be that of the squares of the semi-diameters parallel to the secants (see pa. 19). In the parabola chords c, c' through the focus parallel to the given lines are divided by the focus into parts, of which the rectangles are $\frac{1}{4}pc$ and $\frac{1}{4}pc'$ (135). Hence in the parabola

$$AP \cdot AP' : Ap \cdot Ap' :: c : c'$$

where c, c' are focal chords parallel to AP, Ap . If the points P, P' as also p, p' coincide, the secants become tangents, and the rectangles become the squares of these tangents. Hence tangents from any point to an ellipse or hyperbola are, as the diameters, parallel to them; and the squares of tangents to a parabola from any points are to each other as the focal chords parallel to them.

PROBLEM.

(198.) To determine the general equation of the tangent to a line of the second order.

Let $(x', y'), (x'', y'')$ be two points on the curve, then, for the secant passing through them, we have the equation

$$y - y' = \frac{y' - y''}{x' - x''}(x - x') \dots (1),$$

and from the equation of the curve.

$$\left. \begin{aligned} Ay'^2 + Bx'y' + Cx'^2 + Dy' + Ex' + F &= 0 \\ Ay''^2 + Bx''y'' + Cx''^2 + Dy'' + Ex'' + F &= 0 \end{aligned} \right\} \dots (2).$$

Taking the difference,

$$A(y'^2 - y''^2) + B(x'y' - x''y'') + C(x'^2 - x''^2) + D(y' - y'') + E(x' - x'') = 0.$$

Substituting, in this equation,

$$\begin{aligned} (y' + y'') (y' - y'') & \text{ for } y'^2 - y''^2, \\ (x' + x'') (x' - x'') & \text{ for } x'^2 - x''^2, \\ \text{and } x'(y' - y'') + y'(x' - x'') & \text{ for } x'y' - x''y'', \end{aligned}$$

it becomes

$$(y' - y'') [A (y' + y'') + Bx' + D] + (x' - x'') [C (x' + x'') + By' + E] = 0,$$

from which we obtain

$$\frac{y' - y''}{x' - x''} = - \frac{C (x' + x'') + By' + E}{A (y' + y'') + Bx' + D}.$$

By substituting this value of $\frac{y' - y''}{x' - x''}$, in (1), and then, supposing the points (x', y') , (x'', y'') to coincide, we have for the equation of the tangent passing through (x', y') ,

$$y - y' = - \frac{2Ca' + By' + E}{2Ax' + Bx' + D} (x - x') \dots (3).$$

If the tangent is to pass through a given point (α, β) , without the curve, then, in this equation, we must substitute for the general symbols x and y , the particular values α and β ; and the unknown point of contact, (x', y') , may be determined, analytically, by means of equations (2) and (3), or geometrically, by constructing the loci of these equations.

*M. Puissant** has given a simple and elegant method of arriving at the general equation of the tangent. He refers the curve to *polar* coordinates, assuming the pole on the curve, and then enquires

* Recueil de diverses propositions de Géométrie.

what angle the revolving line must make with the fixed axis, when the radius vector becomes 0, that is, when the line becomes a tangent.

Thus, if it be required to draw a tangent through any point, P, in a line of the second order, represented by the equation

$$y^2 = mx + nx^2 \dots (4),$$

let there be substituted for x and y the values

$$x = x' + r \cos \omega, \quad y = y' + r \sin \omega,$$

or rather, for simplicity,

$$x = x' + rp, \quad y = y' + rq \dots (5),$$

in which x' and y' are the coordinates of the point P, and we shall have the transformed equation

$$(y' + rq)^2 = m(x' + rp) + n(x' + rp)^2 \dots (6),$$

characterizing the proposed curve, when related to polar coordinates, of which the origin is P, and the fixed line parallel to the primitive axis of x .

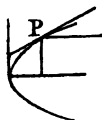
If this equation be developed, and the terms arranged according to the powers of r , the result will evidently be of the form

$$Mr^2 + Br + C = 0.$$

Now it is easy to perceive that the term C, which is independent of r , must represent $y'^2 - mx' - nx'^2$; and this being equal to 0, by equation (4), the transformed equation will be simply

$$Mr + B = 0.$$

When, therefore, $r = 0$, that is, when the radius vector becomes a tangent at the point P, there must exist the condition



$$B = 0;$$

so that, by equating the coefficient of the first power of r , in the development of (6), to 0, we have

$$2y'q - mp - 2npx' = 0,$$

whence

$$\frac{q}{p} = \frac{\sin \omega}{\cos \omega} = \tan \omega = \frac{m + 2nx'}{2y'}$$

This then is the expression for the trigonometrical tangent of the angle formed by the tangent to the curve at P, and the axis of x , and consequently the equation of the tangent is

$$y - \beta = -\frac{2nx' + m}{2y'}(x - \alpha),$$

in which (α, β) denotes any given point in the tangent, and (x', y') the point of contact. If this latter be the given point, the equation is

$$y - y' = \frac{2nx' + m}{2y'}(x - x') \dots (7).$$

The same reasoning, applied to the more general equation,

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0,$$

shews that, after substituting the values (5) for x and y , we need attend in the result only to the coefficient of the first power of r , which being equated to 0, will furnish the value of $\frac{q}{p}$, or $\tan \omega$.

This equation will be

$$2Ayq + B(qx' + py') + 2Cpx' + Dq + Ep = 0,$$

whence

$$\frac{q}{p} = \tan \omega = -\frac{2Cx' + By' + E}{2Ay' + Bx' + D}$$

and therefore the equation of the tangent through the point (x', y') is

$$y - y' = -\frac{2Cx' + By' + E}{2Ay' + Bx' + D}(x - x'),$$

the same as was determined by the first method (3).

By reduction, this equation becomes

$$(2Ay' + Bx' + D)y + (2Cx' + By' + E)x + Dy' + Ex' + 2F = 0,$$

in which may be observed this analogy to the equation of the curve, viz. that, if the accents be effaced, each term in the equation of the curve will appear twice in that of the tangent, from which circumstance the following rule has been contrived for arriving at the equation of the tangent, with the proper accents, by means of the equation of the curve.

Substitute, in the equation of the curve, $x'x$ for x^2 , $y'y$ for y^2 , $x'y$ for xy , and x', y' for x, y .

Repeat the equation thus written, taking care, however, to change x' into x , and x into x' , as also y' into y , and y into y' . The sum of these two equations will be the equation of the tangent.

Thus, let the parabola of which the equation is $y^2 = px$ be proposed, then the two equations to be added will be

$$y'y = px'$$

$$\text{and } y y' = px;$$

therefore $2y'y = p(x' + x)$ is the equation sought.

If we take the general equation $y^2 = mx + nx^2$, we shall have to add the equations

$$y'y = mx' + nx'x$$

$$yy' = mx + nx x';$$

$$\text{therefore } 2y'y = (m + 2nx')x + mx'$$

is the equation of the tangent, and to this form equation (7) may be reduced.

THEOREM.

(199.) If through any given point, chords are drawn to a line of the second order, and tangents be applied to their extremities, these tangents will intersect on a straight line.

Let the coordinates of the given point be a, b , and let (x', y') , (x'', y'') , represent the extremities of either of the chords passing through it; let also (α, β) , be one of the points of intersection. Then for the equation of tangents passing through these points we have, by employing the expression at the conclusion of last problem,

$$2y' \beta = (m + 2nx') \alpha + mx' \dots (1)$$

$$2y'' \beta = (m + 2nx'') \alpha + mx'' \dots (2).$$

Now the equation of the line joining the points of contact of these tangents must be

$$2y\beta = (m + 2nx) \alpha + mx,$$

for this equation must represent some straight line, being of the first degree; and it passes through the two points (x', y') , and (x'', y'') , because equations (1) and (2) subsist; therefore it can represent no other than the *chord of contact*. As (a, b) , is always a point on this chord, we have

$$2b\beta = (m + 2na) \alpha + ma,$$

this equation being of the first degree in α and β shews that the point (α, β) is always on a straight line.

THEOREM.

(200.) If two ellipses, or two hyperbolas have their major diameters related to each other in the same ratio as their minor diameters they will be *similar*; that is, any chord in the one will be to a similarly posited chord in the other in that ratio; the arcs cut off by these chords will also be to each other in the same ratio; and the segments cut off will be to each other as the squares of the terms of the ratio.

Let the principal semi-diameters of one curve be represented by A, B , those of the other by a, b . From the centre of the former let any radius vector R be drawn, and from the centre of the latter let there be drawn the radius vector r similarly posited, that is, inclined at the same angle ω to the major diameter. These radii vectores will be expressed as follows (130, 132),

$$R = A \sqrt{\left\{ \frac{1 - E^2}{1 - E^2 \cos^2 \omega} \right\}}$$

$$r = a \sqrt{\left\{ \frac{1 - e^2}{1 - e^2 \cos^2 \omega} \right\}}$$

in which

$$E = 1 \mp \frac{B^2}{A^2}, \quad e = 1 \mp \frac{b^2}{a^2}.$$

But by the hypothesis $\frac{B}{A} = \frac{b}{a}$; hence $E = e$, and therefore

$$\frac{R}{r} = \frac{A}{a}. \quad \text{Similarly } \frac{R'}{r'} = \frac{A}{a}$$

$$\therefore \frac{R}{r} = \frac{R'}{r'}.$$

R', r' being any other similarly posited radii vectores. The

triangles formed by these radii and lines joining the extremities of R , R' and of r , r' will thus be similar (*Euc.* VI. 6); hence these latter lines, or similarly placed *chords*, are to each other in the fixed ratio.

If the angle between R , R' be divided into any number of equal parts, and the equal one between r , r' into a like number of equal parts, and the extremities of the intermediate radii joined; any one of these joining lines, or chords, will be to the corresponding chord in the other figure in the fixed ratio, so that the sum, or the polygonal line, in one figure will be to that in the other in this same ratio. Moreover, the original chords being the bases of the polygons, and these being obviously similar, they are to each other as the squares of the bases, or as A^2 to a^2 . Now all this is true, however numerous be the sides of the polygonal line spoken of; that is, however near it approaches towards coincidence with the arc in whose extremities it terminates, which arc is the *limit* to its length, and the segment cut off by the original chord is the *limit* to the surface of the polygon. Hence the foregoing conclusions being true up to these limits we have

$$\text{arc } L : \text{arc } l :: A : a$$

and

$$\text{surface } S : \text{surface } s :: A^2 : a^2.$$

It follows that *the perimeters of two similar ellipses are to each other as their like diameters, and their surfaces as the squares of these diameters.*

THEOREM.

(201.) All parabolas are similar figures.

As before, call any two similarly placed radii in two parabolas R and r . Then by the polar equation of the curve we have (133)

$$R = \frac{2M}{1 + \cos \omega}, \quad r = \frac{2m}{1 + \cos \omega}$$

$$\therefore \frac{R}{r} = \frac{M}{m}. \text{ Similarly } \frac{R'}{r'} = \frac{M}{m}$$

$$\therefore \frac{R}{r} = \frac{R'}{r'},$$

R', r' being any other similarly posited radii.

Reasoning now precisely as in the foregoing theorem, we conclude that the chords joining the extremities of R, r and those of R', r' are to each other as M to m , that

$$\text{arc } L : \text{arc } l :: M : m$$

and that

$$\text{surface } S : \text{surface } s :: M^2 : m^2.$$

Every two parabolas are therefore similar figures; the like chords being to each other as the parameters, and the corresponding segments as the squares of the parameters.

Scholium.

It is easy to see that in two similar figures, whatever point be taken in the plane of one there always exists a corresponding point in the plane of the other, such that similarly posited radii from each point to the respective curves will be to each in the same constant ratio. For to any assumed point, P , in the plane of one, let lines be drawn from the extremities of any chord, and from the extremities of a similar chord in the other, let lines be drawn towards the same parts, forming a triangle, whose vertex is P' similar to the former. Then the corresponding sides of these triangles, being to each other as their bases or chords, are in the same constant ratio. And if from the points P, P' new lines be drawn to the extremities of like chords, similar triangles will again be formed, having their like sides in the same ratio. Two points P, P' thus similarly posited in the planes of two similar curves are called *centres of similitude*

THEOREM.

(202.) If a curve of the second order be referred to a system of conjugate axes, and a point in its plane be found, such that its distance from any point whatever in the curve be a rational function of the abscissa of that point, then the point thus found can be no other than a focus of the curve.

First, for the Ellipse.

Let (x', y') denote the fixed point found, then for its distance from any point (x, y) in the curve, we have (15),

$$D^2 = (x - x')^2 + (y - y')^2 + 2(x - x')(y - y') \cos A,$$

where A is the inclination of the axes.

Or, developing this expression, and putting for y its equal

$$\sqrt{B^2 - \frac{B^2 x^2}{A^2}}, \text{ we have}$$

$$D^2 = \frac{A^2 - B^2}{A^2} x^2 - 2x'x + x'^2 + y'^2 + B^2 - 2y'(x - x') \cos A \\ + \{2(x - x') \cos A - 2y'\} \sqrt{B^2 - \frac{B^2 x^2}{A^2}}$$

Now it is obviously impossible that D can be a rational function of x while the irrational function $\sqrt{B^2 - \frac{B^2 x^2}{A^2}}$ remains in this ex-

pression; hence the term in which it enters must disappear, that is, the point (x', y') must be such that

$$2(x - x') \cos A - 2y' = 0$$

for every value of x .

This condition gives the equation

$$x - x' = \frac{y'}{\cos A}.$$

But the first side of this equation is indeterminate, inasmuch as x is; the second side, therefore, must also be indeterminate, although, by hypothesis, y' , and $\cos A$, have certain fixed values; these values, therefore, can be no other than $y' = 0$, and $\cos A = 0$, or $A = 90^\circ$, in which case alone

$$x - x' = \frac{0}{0} = \text{an indeterminate quantity.}$$

It follows, therefore, that the conjugate axes must be the *principal* axes of the curve, and that (since $y' = 0$) the fixed point must be on one of them. With this condition the expression for D^2 becomes

$$D^2 = \frac{A^2 - B^2}{A^2} x^2 - 2x'x + (x'^2 + B^2),$$

and we have now to inquire in what circumstances the root of this square can be a rational function of x .

In order to this, assume $D = bx + a$,

$$\therefore b^2 x^2 + 2bax + a^2 = \frac{A^2 - B^2}{A^2} x^2 - 2x'x + (x'^2 + B^2),$$

then, comparing the coefficients of the like powers of x , we have for the determination of x' the conditions

$$b^2 = \frac{A^2 - B^2}{A^2}, 2ba = -2x', a^2 = x'^2 + B^2.$$

From the first two we get $a^2 = \frac{A^2 x'^2}{A^2 - B^2}$, and this value of a^2 , substituted in the third, gives

$$\frac{A^2 x'^2}{A^2 - B^2} = B^2 + x'^2 \therefore x' = \pm \sqrt{A^2 - B^2} = \pm c.$$

Hence x' has two values that will satisfy the proposed condition, viz. $x = +c$ and $x' = -c$, showing that the points sought are no other than the foci.

The same process applies to the hyperbola when the sign of B^2 is changed.

Second, for the Parabola.

Since, in the parabola, $y = \sqrt{2px}$, therefore

$$D^2 = x^2 - (2x' - 2p)x + x'^2 + y'^2 - 2y'(x - x') \cos A \\ + \{2(x - x') \cos A - 2y'\} \sqrt{px};$$

hence we must conclude, as before, that

$$2(x - x') \cos A - 2y' = 0 \therefore x - x' = \frac{y'}{\cos A} = \frac{0}{0}, \text{ as before,}$$

$$\therefore D^2 = x^2 + 2(p - x')x + x'^2.$$

Put $D^2 = (x + q)^2 = x^2 + 2qx + q^2 = x^2 + 2(p - x')x + x'^2$, then, comparing the coefficients,

$$q = p - x' \text{ and } q^2 = x'^2,$$

$$\therefore q = x', \therefore p = 2x', \therefore x' = \frac{1}{2}p, \text{ the abscissa of the focus.}$$

PROBLEM.

(203.) To determine a cube which shall be double a given cube.

Let the side of the given cube be a , and that of the required cube x , then we are to determine x from the equation

$$x^3 = 2a^3,$$

or

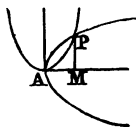
$$x^4 = 2a^3x.$$

Substitute, in this equation, for x^3

$$x^3 = ay \dots (1),$$

and it becomes

$$a^3y^2 = 2a^3x \text{ or } y^2 = 2ax \dots (2).$$



Now equations (1) and (2) represent parabolas referred to the same axes, the parameter of the one being a , and that of the other $2a$. At the point where these parabolas intersect the abscissa will be common to both, therefore this value of x , satisfying both the equations (1) and (2), must also satisfy the proposed, so that the side AM of the required cube may be determined by construction.

PROBLEM.

(204.) To trisect an angle.

By trigonometry, equation 14, p. 50,

$$\cos 3A = \frac{4 \cos A - 3 \cos^3 A}{\cos^3 A}$$

or, putting $\cos 3A = a$, and $\cos A = x$, we have

Q

$$x^3 - \frac{3R^2}{4}x - \frac{R^3a}{4} = 0,$$

from which equation we are to determine the values of x by construction,

Multiplying the terms by x , it becomes

$$x^4 - \frac{3R^2}{4}x^2 - \frac{R^3a}{4}x = 0.$$

In this equation assume

$$x^2 = \frac{1}{2}Ry \dots (1),$$

and it becomes

$$y^2 - \frac{3R}{2}y - ax = 0 \dots (2).$$

If these two equations, which denote parabolas, be constructed in reference to the same axes, the abscissas of their points of intersection will be the three values of the cosine of A to radius R .

These values correspond to the three analytical values

$$\cos A, \cos (\frac{1}{3}\pi + A), \cos (\frac{2}{3}\pi + A),$$

since the given cosine, a , belongs equally to either of the three arcs

$$3A, (2\pi + 3A), (4\pi + 3A).^*$$

* The same cosine equally belongs to the arcs $(6\pi + 3A)$, $(8\pi + 3A)$, &c.; but the cosine of the third part of either of these will always be the same as one of the three cosines in the text; thus $\cos (2\pi + A) = \cos A$, $\cos (2\pi + \frac{1}{3}\pi + A) = \cos (\frac{1}{3}\pi + A)$, &c. Also, since the cosine a will remain the same, although $3A$ be negative, it will remain the same also for every arc in the series, $-3A$, $(2\pi - 3A)$, $(4\pi - 3A)$, $(6\pi - 3A)$, &c.; but for these also the same remarks apply; that is, the cosine of the third part of either will be one of the cosines in the text; thus: $\cos. (\frac{1}{3}\pi - A) = \cos [2\pi - (\frac{1}{3}\pi - A)] = \cos (\frac{1}{3}\pi + A)$, $\cos (\frac{2}{3}\pi - A) = \cos (\frac{2}{3}\pi + A)$ &c.

The last two problems may serve to show the application of curves to the solution of equations. If the equation do not exceed the fourth degree, the roots may always be determined geometrically by the intersections of two lines of the second order. But this mode of determining the roots of equations is never employed in practice; the most accurate as well as most expeditious process being that of numerical approximation. The best method of approximating to the roots of equations is that discovered by *Mr. Horner*, and printed in the *Philosophical Transactions* for 1819; for a full account of which see the treatise on the *Theory of Equations*, by the author of the present work, who, by simplifying the investigation, and by modifying a little the numerical operation, has endeavoured to render *Mr. Horner's* method readily intelligible to the Algebraical Student.

For further applications of the theory of curves to the construction of equations the student is referred to *Bourdon*, *Application de l'Algèbre à la Géométrie*; *Lacroix*, *Trigonométrie*; *Garnier*, *Analyse Algébrique*, and *Lardner's Algebraic Geometry*; sections xx. and xxi.

THEOREM.

(205.) Five points being given on a plane, of which no three are situated on the same straight line, it is possible to describe a line of the second order which shall pass through them all.

For, let the general equation

$$Ay^2 + Bxy + Cx^2 + Dy + Ex = F \dots (1)$$

be divided by *A*, and it will then assume the form

$$y^2 + bxy + cx^2 + dy + ex = f \dots (2),$$

so that the equation of the second degree, in its most general form, contains five coefficients, *b*, *c*, *d*, *e*, and *f*; the values of which may be arbitrarily assumed; they may, therefore, be so determined as to subject the curve, into whose equation they enter, to pass through the points (*a*, *β*), (*a'*, *β'*), (*a''*, *β''*), (*a'''*, *β'''*), (*a''''*, *β''''*), since we

shall have, for this purpose, the five simple equations

$$\beta^2 + b\alpha \beta + c\alpha^2 + d\beta + e\alpha = f$$

$$\beta'^2 + b\alpha' \beta' + c\alpha'^2 + d\beta' + e\alpha' = f$$

$$\beta''^2 + b\alpha'' \beta'' + c\alpha''^2 + d\beta'' + e\alpha'' = f$$

$$\beta'''^2 + b\alpha''' \beta''' + c\alpha'''^2 + d\beta''' + e\alpha''' = f$$

$$\beta''''^2 + b\alpha'''' \beta'''' + c\alpha''''^2 + d\beta'''' + e\alpha'''' = f$$

from which the values of the unknowns b, c, d, e , and f , may obviously be determined, and these values substituted in equation (2) will render that equation the representative of the required curve. If we are not restricted in the choice of axes of coordinates, they may be so assumed as to render some of the preceding equations of condition of simpler form. Thus, by taking one of the points, as (α, β) , for the origin, we shall have, for the first equation, merely $0 = f$; and if each axis be drawn through a separate point, as (α', β') , and (α'', β'') , the next two equations will be $\beta'^2 + d\beta' = f$, and $c\alpha''^2 + e\alpha'' = f$. If the curve sought ought to be a parabola, only four arbitrary points must be assumed, because, in the equation of this curve, only four of the five coefficients are arbitrary, since between the two, b and c , there must exist the relation

$$b^2 - 4c = 0;$$

hence this equation, combined with those *four* of the preceding equations into which the given coordinates enter, will fix the values of the five coefficients sought; and as here b has two values, b positive and b negative, for the same value of c , there may be two parabolas passing through the same four points. But no curve of the second order can intersect another in more points than four, since the coefficients determined by the five preceding equations admit each of but one value.

If the points through which the conic section is to pass be posited, so that more than two of them lie in the same straight line, then the final equation will obviously represent, not a curve, but one of the varieties—a system of straight lines.

PROBLEM.

(206.) To determine a curve which shall pass through any proposed number of given points.

Let us represent the given points by

$$(\alpha, \beta), (\alpha', \beta'), (\alpha'', \beta''), (\alpha''', \beta'''), (\alpha''', \beta'''), \&c.$$

then it is obvious that several curves might be described passing through these points, some expressible by equations, and others not. But it is desirable to know which of all the possible curves is the simplest, or admits of the easiest description. Now, those curves are most easily described of which any ordinate is a rational and integral function of the abscissa, because the value of the ordinate corresponding to any assumed abscissa will, in this case, never be encumbered with radicals. These curves are included in the equation

$$y = A + Bx + Cx^2 + Dx^3 + \&c.$$

and they are called *parabolic curves*, because the parabola, of which the equation is $y = A + Bx + Cx^2$, is obviously one of them. The order of the curve depends upon the highest power of x ; the common parabola is of the second order, and that in which x^3 is the highest power of x is of the third order, and so on.

Hence, if we take the parabola whose order is one below the number of proposed points, we shall have to determine the same number of coefficients, $A, B, C, \&c.$, from the simple equations

$$\left. \begin{aligned} \beta &= A + B\alpha + C\alpha^2 + D\alpha^3 + \&c. \\ \beta' &= A + B\alpha' + C\alpha'^2 + D\alpha'^3 + \&c. \\ \beta'' &= A + B\alpha'' + C\alpha''^2 + D\alpha''^3 + \&c. \\ \beta''' &= A + B\alpha''' + C\alpha'''^2 + D\alpha'''^3 + \&c. \\ &\&c. \quad \&c. \quad \&c. \end{aligned} \right\} \dots\dots (1).$$

To do this in the simplest manner, let each equation be subtracted from the next, and we shall have

$$\begin{aligned}
 (\beta' - \beta) &= B(\alpha' - \alpha) + C(\alpha'^2 - \alpha^2) + D(\alpha'^3 - \alpha^3) + \&c. \\
 (\beta' - \beta'') &= B(\alpha'' - \alpha') + C(\alpha''^2 - \alpha'^2) + D(\alpha''^3 - \alpha'^3) + \&c. \\
 (\beta''' - \beta'') &= B(\alpha''' - \alpha'') + C(\alpha'''^2 - \alpha''^2) + D(\alpha'''^3 - \alpha''^3) + \&c.
 \end{aligned}$$

$\&c.$ $\&c.$ $\&c.$

and, by division,

$$\begin{aligned}
 \frac{\beta' - \beta}{\alpha' - \alpha} &= B + C(\alpha' + \alpha) + D(\alpha'^2 + \alpha'\alpha + \alpha^2) + \&c. = a \\
 \frac{\beta' - \beta''}{\alpha' - \alpha'} &= B + C(\alpha'' + \alpha') + D(\alpha''^2 + \alpha''\alpha' + \alpha'^2) + \&c. = a' \\
 \frac{\beta''' - \beta''}{\alpha''' - \alpha''} &= B + C(\alpha''' + \alpha'') + D(\alpha'''^2 + \alpha''' \alpha'' + \alpha''^2) + \&c. = a''
 \end{aligned}
 \left. \vphantom{\begin{aligned} \frac{\beta' - \beta}{\alpha' - \alpha} \\ \frac{\beta' - \beta''}{\alpha' - \alpha'} \\ \frac{\beta''' - \beta''}{\alpha''' - \alpha''} \end{aligned}} \right\} (2).$$

$\&c.$ $\&c.$ $\&c.$

The values a, a', a'' , are known, because the first member of each equation is known.

Now these equations are of the same form as those originally proposed, and A is found eliminated; hence, by performing a similar process with these equations as with the first group, we shall have

$$\begin{aligned}
 \frac{a' - a}{\alpha'' - \alpha} &= C + D(\alpha'' + \alpha' + \alpha) + \&c. = b \\
 \frac{a'' - a'}{\alpha''' - \alpha'} &= C + D(\alpha''' + \alpha'' + \alpha') + \&c. = b'
 \end{aligned}
 \left. \vphantom{\begin{aligned} \frac{a' - a}{\alpha'' - \alpha} \\ \frac{a'' - a'}{\alpha''' - \alpha'} \end{aligned}} \right\} \dots (3).$$

$\&c.$ $\&c.$ $\&c.$

In these equations, the next coefficient, B , is eliminated, and the values of $b, b', \&c.$ are known, because the first member of each equation is known. By continuing the process, we shall eliminate the coefficients one by one, till we come to the last, the value of which may then be determined from the final simple equation, and thence all the others.

Suppose, for example, only three points (α, β) , (α', β') , and (α'', β'') , are proposed, then the equation of the curve is

$$y = A + Bx + Cx^2,$$

and the equations (2), (3), become

$$\frac{\beta' - \beta}{\alpha' - \alpha} = B + C(\alpha' + \alpha) = a$$

$$\frac{\beta'' - \beta'}{\alpha'' - \alpha'} = B + C(\alpha'' + \alpha') = a'$$

$$\frac{\alpha' - \alpha}{\alpha'' - \alpha} = C = b$$

Substituting this value of C , in the first equation, we have

$$B = a - b(\alpha' + \alpha).$$

Also, since from the proposed equation we have

$$\beta = A + B\alpha + C\alpha^2,$$

we obtain for A , after having replaced B and C by the values just determined, the expression

$$A = \beta - a\alpha + b\alpha\alpha'.$$

Having thus determined the values of the three coefficients, we have for the equation of the required parabola,

$$y = \beta - a\alpha + b\alpha\alpha' + (a - b\alpha' - b\alpha)x + bx^2,$$

or

$$y = \beta + a(x - \alpha) + b(x - \alpha)(x - \alpha').$$

Lagrange, after having given this solution from *Newton*, observes, that it may be much more simply obtained, from the following considerations:

Since y ought to become $\beta, \beta', \beta'' \dots$ when x becomes $a, a', a'' \dots$, it is obvious that the expression for y must be of the form

$$y = A'\beta + B'\beta' + C'\beta'' + \dots,$$

where the quantities $A', B', C', \&c.$ must be functions of x , such that, when we put

$$x = a, \quad \text{we must have} \quad A' = 1, B' = 0, C' = 0 \dots$$

$$x = a', \quad A' = 0, B' = 1, C' = 0 \dots$$

$$x = a'', \quad A' = 0, B' = 0, C' = 1 \dots$$

&c.

&c.

&c.

consequently, the values of $A', B', C', \&c.$ must necessarily take the form

$$A' = \frac{(x-a')(x-a'')(x-a''') \dots}{(a-a')(a-a'')(a-a''') \dots}$$

$$B' = \frac{(x-a)(x-a''')(x-a''') \dots}{(a'-a)(a'-a'')(a'-a''') \dots}$$

$$C' = \frac{(x-a)(x-a')(x-a''') \dots}{(a''-a)(a''-a')(a''-a''') \dots}$$

&c.

&c.

where the number of factors in each numerator and denominator is one less than the number of given points. Hence the general expression for y is

$$\begin{aligned} y = & \frac{(x-a')(x-a'')(x-a''') \dots}{(a-a')(a-a'')(a-a''') \dots} \beta \\ & + \frac{(x-a)(x-a''')(x-a''') \dots}{(a'-a)(a'-a'')(a'-a''') \dots} \beta' \\ & + \frac{(x-a)(x-a')(x-a''') \dots}{(a''-a)(a''-a')(a''-a''') \dots} \beta'' \\ & \quad \quad \quad \&c. \quad \quad \quad \&c. \end{aligned}$$

Newton's method gives for the general expression for y ,

$$y = \beta + a(x - a) + b(x - a)(x - a') + c(x - a)(x - a')(x - a'') + \&c.$$

These two expressions are different only in form, as may be ascertained by developing the values of a , b , c , &c. and arranging the terms according to the quantities β , β' , β'' , &c.

Either of the preceding values of y may be considered as a solution to this problem, viz. To determine the general relation which exists between two variable quantities, x , y , from knowing the relation which exists in the particular cases $x = a$, $y = \beta$; $x = a'$, $y = \beta'$; $x = a''$, $y = \beta''$; &c.

This is an important problem, being the foundation of the *method of interpolation*, since it enables us, from having a certain number of terms of a series given, the law of which is not known, to arrive at an approximate expression for the general term of that series, and thence to interpolate between the given terms as many more as we please, all governed by the same law. Of the two general expressions for this purpose just given, the former is the more commodious in calculation, because the several terms may be computed by logarithms. Nevertheless, the latter expression leads to a very neat and commodious formula, when we suppose the quantities a , a' , a'' , &c. to be in arithmetical progression, as is generally the case in practice.

Let h be the common difference of the progression, then

$$a' = a + h, a'' = a + 2h, a''' = a + 3h, \&c.$$

let also $x = a + h'$, then

$$x - a = h', x - a' = h' - h, x - a'' = h' - 2h, \&c.$$

Now, putting, for brevity, $\Delta\beta$, $\Delta\beta'$, $\Delta\beta''$, &c. for the several differences, $\beta' - \beta$, $\beta'' - \beta'$, $\beta''' - \beta''$, &c. we have (2)

$$a = \frac{\Delta\beta}{h}, a' = \frac{\Delta\beta'}{h}, a'' = \frac{\Delta\beta''}{h}, \&c.$$

putting, in like manner, $\Delta^2\beta$, $\Delta^2\beta'$, &c. for the second differences, $\Delta\beta' - \Delta\beta$, $\Delta\beta'' - \Delta\beta'$, &c. we have

$$b = \frac{\Delta^2\beta}{1 \cdot 2h^2}, \quad b' = \frac{\Delta^2\beta'}{1 \cdot 2h^2}, \text{ \&c.}$$

substituting also $\Delta^3\beta$, &c. for the third differences, $\Delta^2\beta' - \Delta^2\beta$, &c. there results

$$c = \frac{\Delta^3\beta}{1 \cdot 2 \cdot 3h^3}, \text{ \&c.}$$

therefore the formula becomes

$$y = \beta + \frac{h'}{h} \Delta\beta + \frac{h'(h'-h)}{h \cdot 2h} \Delta^2\beta + \frac{h'(h'-h)(h'-2h)}{h \cdot 2h \cdot 3h} \Delta^3\beta + \dots$$

which expression for y terminates, when the differences at length become constant, and finally vanish. In such a case the value of y is perfectly accurate. But when the differences only tend towards zero, without actually reaching it, then the value of y must be regarded as an approximation only, differing the less from the truth, the higher the order of the differences at which we stop. The inaccuracy in this case is due to the error we commit in rejecting the subsequent orders of differences, as if that immediately beyond the one at which we stop were actually zero.

As an application of this formula, the following example is frequently given.

EXAMPLE.

To compute the logarithm of the number 3.1415926536 by means of a table of logarithms from 1 to 1000, calculated to ten places of decimals.

Regarding the logarithms in the table as the particular values of y , the corresponding numbers being the values of x , we shall have,

by taking the successive differences, the following values

$$\begin{array}{lcl}
 \beta & = \log 3 \cdot 14 = \cdot 4969296481 & \Delta \beta = \cdot 13809057 \\
 \beta' & = \log 3 \cdot 15 = \cdot 4983105538 & \Delta \beta' = \cdot 13765258 \\
 \beta'' & = \log 3 \cdot 16 = \cdot 4996870826 & \Delta \beta'' = \cdot 13721796 \\
 \beta''' & = \log 3 \cdot 17 = \cdot 5010592632 & \Delta \beta''' = \cdot 13678578 \\
 \beta'''' & = \log 3 \cdot 18 = \cdot 5024271200 & \\
 \\
 \Delta^2 \beta & = -43769 & \Delta^2 \beta = 277 \\
 \Delta^2 \beta' & = -43492 & \Delta^2 \beta' = 274 \\
 \Delta^2 \beta'' & = -43218 & \Delta^2 \beta'' = 274
 \end{array}
 \left| \begin{array}{l} \\ \\ \\ \Delta^4 \beta = -3; \end{array} \right.$$

consequently,

$$\begin{aligned}
 \Delta \beta &= \cdot 0013809057, \quad \Delta^2 \beta = -\cdot 0000043769 \\
 \Delta^3 \beta &= \cdot 0000000277, \quad \Delta^4 \beta = -\cdot 0000000003.
 \end{aligned}$$

The fourth difference being so small, it is plain that the fifth may be rejected without affecting the accuracy of the value of y , within the limits of the first ten decimals.

Now the constant difference, h , of the progression is $\cdot 01$, and $3 \cdot 14$ being taken for a , and $3 \cdot 1415926536$ for x , we have $h' = x - a = \cdot 0015926536$; hence

$$\begin{aligned}
 \frac{h'}{h} &= \cdot 15926536 \\
 \frac{h' - h}{2h} &= \frac{h'}{2h} - \frac{1}{2} = -\cdot 42036732 \\
 \frac{h' - 2h}{3h} &= \frac{h'}{3h} - \frac{2}{3} = -\cdot 61357821 \\
 \frac{h' - 3h}{4h} &= \frac{h'}{4h} - \frac{3}{4} = -\cdot 71018386.
 \end{aligned}$$

These values, substituted in the formula

$$y = \beta + \frac{h'}{h} \Delta \beta + \frac{h'(h' - h)}{h \cdot 2h} \Delta^2 \beta +$$

$$\frac{h'(h' - h)(h' - 2h)}{h \cdot 2h \cdot 3h} \Delta^3 \beta + \frac{h'(h' - h)(h' - 2h)(h' - 3h)}{h \cdot 2h \cdot 3h \cdot 4h} \Delta^4 \beta,$$

give

$$y = \log 3 \cdot 1415926536 = \cdot 4971498726.$$

(207.) We shall give one more example of the application of the formula of interpolation, which may serve as a specimen of its use in Practical Astronomy.

Given five places of a comet, as follows :

α	=	November 5	at 8h. 17m.,	in Cancer	$2^\circ 30' = 150' = \beta$
α'	= 6	$4^\circ 7' = 247' = \beta'$
α''	= 7	$6^\circ 20' = 380' = \beta''$
α'''	= 8	$9^\circ 10' = 550' = \beta'''$
α''''	= 9	$12^\circ 40' = 760' = \beta''''$

Required its place on the 7th at 14h. 17m.

Here $\alpha = 5$ d. 8h. 17m., $h = 1$ day, $x = 7$ d. 14h. 17m., and $h' = x - \alpha = 2$ d. 6h. = $2 \cdot 25$; also, taking the differences of the series $\alpha, \alpha', \alpha'', \alpha''', \alpha''''$ we have

150 247 380 550 760

97 133 170 210

36 37 40

1 3

2

so that $\Delta \beta = 97$, $\Delta^2 \beta = 36$, $\Delta^3 \beta = 1$, $\Delta^4 \beta = 2$. Hence the approximate value of y is

$$y = 150 + 97 \times 2.25 + \frac{36}{2} \times 2.25 \times 1.25 + \frac{1}{2.3} \times 2.25 \times 1.25 \times .25 +$$

$$\frac{2}{2.3.4} \times 2.25 \times 1.25 \times .25 \times -.75 = 418.96 = 6^{\circ} 58' 57''$$

the place required.

Most of the Tables employed in Practical Astronomy, such as those given in the Nautical Almanac, &c. are constructed by the aid of interpolations. The phenomena are directly calculated for certain epochs; and those which correspond to the intermediate times are deduced from the calculated results, by *differencing* them, as above, till a difference is obtained, which, on account of its smallness, may be rejected without introducing sensible error.

For further applications of the method of Interpolation, see the *Essay on the Construction of Logarithms*, second edition.

GENERAL SCHOLIUM.

(208.) Having discussed pretty fully the various properties of curves of the second order, it remains to make a few general remarks upon the higher orders of curves, or those of which the equations extend beyond the second degree.

If we have an equation of the n th degree, containing two variables, and which is not compounded of equations of inferior degrees, the curve which it represents is said to be of the n th order. But, if the equation is compounded of others of inferior degrees, then also its geometrical representation comprehends all the curves represented by the component equations. Such an assemblage of lines is called a complex line. For instance, the locus of the cubic equation

$$y^3 - axy^2 + bxy - abx^2 - cy + acx = 0,$$

which arises from the multiplication of the two equations

$$y - ax = 0 \dots (1)$$

and

$$y^2 + bx - c = 0 \dots (2),$$

is not a *simple* line of the third order, but a *complex* line, consisting of the straight line represented by the equation (1), and the parabola represented by equation (2). For the coordinates (x, y) of every point in the straight line rendering the factor $y - ax$ equal to 0, the same coordinates must render 0 the product $(y - ax)(y^2 + bx - c)$, that is, they must always satisfy the proposed equation; hence this straight line must belong to the locus, and in the same manner is it shown that the parabola must also belong to the locus.

It appears, therefore, that for an equation of the n th degree to represent a curve of the n th order, it must be such that, when all the terms are arranged on one side, it may not admit of being resolved into rational factors.

The most general form of an equation between two variables of any proposed degree is that which, besides constant quantities, contains every possible combination of the variables, under the condition, that, wherever their product enters, the sum of their exponents shall not exceed the required degree. Thus the most general form of the equation of the third degree is

$$Ay^3 + Ry^2x + Cyx^2 + Dx^3 + Ey^2 + Fyx + Gx^2 + Hy + Kx + L = 0,$$

that of the fourth degree,

$$\left. \begin{aligned} & Ay^4 + Ry^3x + Cy^2x^2 + Dyx^3 + Ex^4 \\ & + Fy^3 + Gy^2x + Hyx^2 + Kx^3 \\ & + Ly^2 + Myx + Nx^2 \\ & + Py + Qx \\ & + R \end{aligned} \right\} = 0;$$

8zc., so that the number of terms in a general equation of the n th degree will be equal to those in

$$(y+x)^n + (y+x)^{n-1} + (y+x)^{n-2} + \dots + (y+x)^0.$$

Now the expansion of any power of a binomial consists of as many terms as there are units in its exponent, and one more (*Alg.* p. 169); hence the sum of the terms in the above series of expansions is that of the arithmetical progression

$$1 + 2 + 3 + 4 + \dots + n + 1 = \frac{1}{2}(n+1)(n+2).$$

We infer, therefore, that, in the general equation of the n th degree, there are $\frac{1}{2}(n+1)(n+2)$ terms, and, consequently, the same number of constant coefficients; we may, however, without diminishing the generality of an equation, divide all its terms by the coefficient of any one of them, and thus reduce the number of arbitrary coefficients to

$$\frac{1}{2}(n+1)(n+2) - 1 = \frac{1}{2}n(n+3).$$

It follows from this, that a curve of the n th order may be made to pass through $\frac{1}{2}n(n+3)$, points arbitrarily assumed; for the coordinates of each point being successively substituted for x, y , in the general equation, will give rise to $\frac{1}{2}n(n+3)$, equations in which the general coefficients are the unknown quantities, and which these equations are sufficient to determine; the values of the coefficients being thus ascertained, the locus of the equation will pass through the proposed points; but if these are so assumed as to render it impossible for any simple curve of the proposed order to pass through them, then the locus determined as above will be a *complex line* of the proposed degree. If the points are all in the same straight line, the equation of the locus will be found to be reducible to the form $(y+ax+b)^n = 0$, which represents n coinciding straight lines.

Let us now inquire in how many points it is possible for a straight

line to intersect a curve of the n th order. Taking the general equation of the n th order, and putting $y = 0$, we have the equation

$$A'x^n + B'x^{n-1} + C'x^{n-2} + \dots + Px + Q = 0,$$

the roots of which are the values of so many abscissas of the points where the axis of x cuts the curve. As the values of the coefficients A' , B' , &c. are quite arbitrary, they may obviously be such as to render these n roots all possible and different from each other; hence a curve of the n th order *may* be cut by a straight line in n points, but not in more; there are, however, not *necessarily* n points of intersection; the number may be less, but cannot be more. If the term $A'x^n$ be absent from the equation of a curve of the n th order, or can by any transformation be removed, then there can at most be but $n - 1$ points of intersection between the curve and axis of x ; if the term $B'x^{n-1}$ be also absent, then the number of intersections will be but $n - 2$, and so on

When any particular curve of the n th order is proposed, then the coefficients A' , B' , &c. become fixed, and the number of intersections will be n , or $n - 2$, or $n - 4$, &c. according as the equation has n , or $n - 2$, or $n - 4$, &c. possible roots.*

Between the curves of the second order and those of the higher orders there exists a very intimate analogy. We can adduce here only the two following instances:

1. If two straight lines, parallel to the axis of y , drawn in a curve of the n th order, be cut by the axis of x , so that the sum of the ordinates on one side be in each case equal to the sum of the

* If any of the roots of this equation are equal, it will intimate that the straight line passes through a *singular* point of the curve. Thus, if two roots are equal, the corresponding point will be either a point of contact or a double point; that is, a point in which two branches of the curve intersect. If three roots be equal, the corresponding point will be either a point of *inflection*, or a triple point, &c. The investigation of the singular points of curves belongs more properly to the *Differential Calculus*. See the treatise on that subject, Chapter IV.

ordinates on the other side, then every other line parallel to these will be cut by the axis in the same manner.

Let the equation of the curve be

$$y^n + (ax + b)y^{n-1} + \dots = 0,$$

and those of the two parallels, $x = p$, and $x = q$, then we have

$$y^n + (ap + b)y^{n-1} + \&c. = 0$$

$$y^n + (aq + b)y^{n-1} + \&c. = 0,$$

and, since in each case the sum of the negative ordinates is equal to the sum of the positive, we have, by the theory of Equations,

$$ap + b = 0, \quad aq + b = 0 \therefore a(p - q) = 0 \therefore a = 0;$$

hence $b = 0$; consequently, whatever be the value of x , we must always have

$$ax + b = 0,$$

which establishes the proposition.

The line thus dividing parallel ordinates is called a *diameter*,

2. If to any line of the n th order two secants, parallel to the sides of a given angle, be drawn, then the continued products of the parts, intercepted between their point of intersection and the curve, will have a constant ratio.

For, taking these two secants as axes, and putting successively $y = 0$, and $x = 0$, the equation gives

$$A'x^n + B'x^{n-1} + C'x^{n-2} + \dots + P'x + Q = 0$$

$$Ay^n + By^{n-1} + Cy^{n-2} + \dots + Px + Q = 0.$$

The roots of these equations give the parts of the two secants intercepted between their intersection, that is, the origin, and the curve.

Hence, dividing the first equation by A' , we have for the product

of its roots, or of the parts of one secant, the expression $\frac{Q}{A'}$.^{*} In like manner, for the product of the parts of the other secant we have the expression $\frac{Q}{A}$. Hence these products are to each other as $\frac{Q}{A'} : \frac{Q}{A}$, or as $A : A'$, that is, they have a constant ratio.

For further particulars respecting the higher curves the student is referred to the comprehensive summary given by *Dr. Gregory*, in the second volume of *Hutton's Mathematics*; to *Lardner's Algebraic Geometry*, section xxi.; or to *Maclaurin, on the General Properties of Geometrical Lines*.

* Theory of Equations, page 15.

SECTION II.

THE POINT, STRAIGHT LINE, AND PLANE, IN SPACE.

(209). THE preceding portion of the present treatise has been occupied in discussing the properties of plane curves; that is to say, of lines of which all the points are situated in the same plane. In these inquiries we have found nothing more to be necessary than to assume, in the same plane with the curve proposed, two fixed lines, or axes, and then to investigate the analytical expression which must characterize the position of every point in the curve, relatively to these assumed axes. Such representation of the proposed curve contains implicitly all its properties, these being severally evolved upon analyzing the characteristic equation. In this more advanced department of our subject, called *Analytical Geometry of three dimensions*, we propose to extend our inquiries to the consideration of lines and surfaces not entirely situated in one plane, and where it will be necessary to employ three axes of reference, instead of two. We shall begin by determining the equation of a point situated in space.

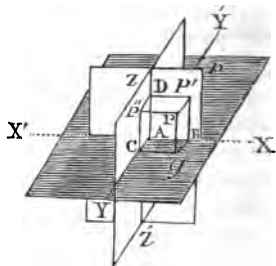
CHAPTER I.

ON THE POINT AND STRAIGHT LINE SITUATED IN SPACE.

Equations of a Point.

(210.) Let ZAX , ZAY , YAX , be three planes which, for simplicity, we shall suppose to intersect at right angles, so that the line ZA will be perpendicular to both AX and AY . Let, also, P be any point in space, whose position it is required to determine relatively to the axes, AX , AY , AZ .

Upon each of the assumed planes let fall, from the point P , the perpendiculars Pp , Pp' , Pp'' ; then, if the lengths of these perpendiculars be given, the position of the point P will be determined. For, conceive the planes PC , PB , PD , to be drawn, forming with the planes DC , DB , CB , the rectangular parallelopiped AP ; then $AD = Pp$,



$AC = Pp'$, and $AB = Pp''$, consequently the points B , C , D , are at given distances from the point A ; if, therefore, through these points three planes parallel to the assumed planes be drawn, the point P will be that in which they all intersect.

The three planes, ZAX , ZAY , YAX , in reference to which the position of the point has been determined, are called the *coordinate planes*; their intersections, AX , AY , AZ , are the *axes of coordinates*, and the distances AB , AC , AD , of the three planes, parallel to the former, from the origin, A , are the three *coordinates* of the

point P, where they intersect. The coordinates of any point are generally denoted by x, y , and z ; if these are known, the point, as we have just seen, is determinable; hence the equations

$$x = a, y = b, z = c$$

are the equations of a point in space.

If the three coordinate planes be produced beyond their intersections, there will obviously be formed about the point A eight triedral angles,* four above the horizontal plane YAX, and four below; hence, to express analytically in which of these angles the proposed point is situated, we must prefix to its ordinates the signs which they must take from considering the axes of ordinates as positive, in one direction, and consequently as negative, in the opposite direction; thus, regarding the axes as positive in the directions AX, AY, AZ, they will be negative in the opposite directions, AX', AY', AZ'; hence we shall have the following variations of the signs of the coordinates for every possible position of the point P.

If $x = +a, y = +b, z = +c$, the point is in the angle AXYZ,

$x = -a, y = +b, z = +c,$	AX'YZ,
$x = +a, y = -b, z = +c,$	AXY'Z,
$x = +a, y = +b, z = -c,$	AXYZ',
$x = -a, y = -b, z = +c,$	AX'Y'Z,
$x = -a, y = +b, z = -c,$	AX'YZ',
$x = +a, y = -b, z = -c,$	AXY'Z',
$x = -a, y = -b, z = -c,$	AX'Y'Z'.

(211.) Of the three coordinate planes that which contains the

* Angles formed by the meeting of three planes in a point.

axes AX, AY, and which is generally the horizontal plane, is called the *plane of xy*; that which contains the axes AX, AZ, is called the *plane of xz*; and the third, containing the axes AY, AZ, is called the *plane of yz*.

If the proposed point be situated in the plane of *xy*, then its distance, *z*, from this plane being 0, its equations will be

$$x = a, y = b, z = 0.$$

If it be on the axis of *x*, that is, on the intersection of the planes of *xy* and *xz*, then its distance from each of these planes being 0, its position will be expressed by the equations

$$x = a, y = 0, z = 0.$$

But if it be at the origin, that is, at the common intersection of the three planes, then, its distance from each being 0, the equations of the point are

$$x = 0, y = 0, z = 0.$$

In like manner, if the point be situated in the plane of *xz*, its equations are

$$x = a, y = 0, z = c;$$

and if it be on the axis of *x*, or on the axis of *z*, we have, respectively

$$x = a, y = 0, z = 0,$$

$$x = 0, y = 0, z = c.$$

Lastly, if the point be in the plane of *yz*, its equations are

$$x = 0, y = b, z = c.$$

(212.) The points *p, p', p''*, where the perpendiculars from P meet the coordinate planes, are called the *projections* of P, on these planes. If the position of any two of these projections were given,

it would be sufficient to determine the point P; for a perpendicular, from either projection to the plane in which it is, necessarily passes through the point P, so that P will be at the intersection of two such perpendiculars; knowing, therefore, two projections, we can always, if required, determine the third.

Suppose, for instance, the projections p, p' , on the planes of xy and xz be known, or, which is the same thing, that we have given the equations of these points, viz.

$$x = a, y = b,$$

$$x = a, z = c:$$

these two equations give for the third projection, p'' , the equations

$$y = b, z = c,$$

and any two of these combined give the equations of P, viz.

$$x = a, y = b, z = c$$

If the coordinate planes had been oblique, instead of rectangular, the preceding equations would have been the same; but the coordinates a, b, c , would then have been oblique, and the projections of P would have been given by lines drawn from P, parallel to the coordinate planes. To distinguish the former from these, they are called the *orthogonal* projections of the point.

On the Equations of the Straight Line in Space.

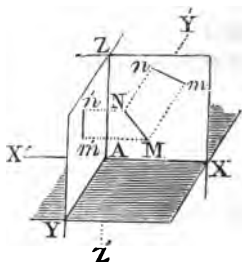
(213.) If, through any given straight line situated in space, a plane, perpendicular to either of the coordinate planes, be drawn, the intersection of the two planes is called the *projection* of the proposed line. The plane thus drawn is called the *projecting plane*; there are, therefore, three projecting planes, each of which

contains the proposed line, and one of its projections; consequently, knowing two of the projections, we may draw two of the projecting planes; and, since the proposed line must be situated in each, their intersection will determine it; hence, in the straight line, as in the point, two projections are sufficient to determine it.

Let MN be a straight line in space, of which the projections on the planes of zx and of zy are mn and $m'n'$, and let the equations of these projections be

$$x = az + \alpha \dots (1),$$

$$y = bz + \beta \dots (2).$$



Assume any point in the projecting plane, Nm , and through it draw in this plane a parallel to AY , then every point in this parallel being equally distant from the plane of zy , and also equi-distant from the plane of xy , it follows that for every point in this line the coordinates x, z , are the same, and one of the points is in the line mn ; but the coordinates x, z , for every point in mn are related, as in equation (1); hence also the coordinates x, z , of every point in the projecting plane, Nm , are related, as in equation (1). In a similar manner, the coordinates y, z , of any point in the line $m'n'$ are the same as those of any point in the projecting plane, Nm' . Hence, at the intersection, MN , of these planes, both the relations (1) and (2) must exist, so that these equations which, taken separately, characterize the two projections, represent, when taken together, the proposed line; therefore the two simultaneous equations

$$\left. \begin{aligned} x &= az + \alpha \\ y &= bz + \beta \end{aligned} \right\}$$

are the equations of the straight line in space.

Hence any assumed value being given to z , one of the coordinates, these equations will make known the other two, and thus the three coordinates of any point in the line may be obtained.

(214.) We have here supposed that the proposed line is projected on the two vertical planes, ZX, ZY; if, however, one had been the horizontal plane, XY, then the equation of the projection on this plane would have exhibited the relation between the coordinates, x, y , of any point in the proposed line; and this, combined with either of the equations, (1), (2), would have equally characterized the proposed line. But the relation between x and y , or the equation of the projection on XY, is readily obtained by eliminating z , in equations (1), (2); this elimination gives the relation

$$y = \frac{b}{a} x - \frac{bx - a\beta}{a}$$

which is, therefore, the equation of the projection on the plane of xy ; and in a similar manner may either projection be obtained from knowing the other two. But the projections usually employed are those on the vertical planes represented by equations (1), (2), in which the vertical axis, AZ, is considered as the axis of abscissas; the horizontal axis, AX, as the axis of ordinates for the projections on the plane of xz ; and the horizontal axis, AY, as the axis of ordinates for the projections on the plane of yz .

The constants, a and b , denote the tangents of the angles which the projections on the vertical planes make with the axis of x ; and α, β , express the distances of the origin from the points where these projections intersect the axes of x and of y , respectively.

PROBLEM I.

(215.) To determine the points where the coordinate planes are pierced by a given straight line.

Let the given straight line be represented by the equations

$$\left. \begin{aligned} x &= az + \alpha \\ y &= bz + \beta \end{aligned} \right\}$$

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Then at the point where this line pierces the plane of xy , $z=0$; substituting therefore this value of z , in the above equations, we obtain for the coordinates, x, y , of the same point

$$x=\alpha, y=\beta.$$

At the point where the line pierces the plane of xz , $y=0$; putting, therefore, this value of y , in the second equation, and substituting the resulting expression for z , in the first, we have for the coordinates, x, z , of this point

$$x = \frac{b\alpha - a\beta}{b}, z = -\frac{\beta}{b},$$

and lastly at the point where the line pierces the plane of yz , $x=0$; putting, therefore, this value of x , in the first equation, and substituting the resulting value of z , in the second, we obtain for the coordinates, y, z , of the same point

$$y = \frac{a\beta - b\alpha}{a}, z = -\frac{\alpha}{a}.$$

The points where a straight line in space pierces the coordinate planes are called the *traces* of the line on those planes.

PROBLEM II.

(216.) To find the equations of a straight line passing through a given point.

Let x', y', z' be the coordinates of the given point, and let the equations of the line sought be

$$\left. \begin{aligned} x &= az + \alpha \\ y &= bz + \beta \end{aligned} \right\} \dots (1).$$

then we must have the conditions

$$x' = ax' + a$$

$$y' = bz' + \beta,$$

which give for a and β the values

$$a = x' - ax' \text{ and } \beta = y' - bz';$$

hence, by substitution, in equations (1), we have

$$\left. \begin{aligned} x - x' &= a(x - x') \\ y - y' &= b(x - x') \end{aligned} \right\} \dots (2),$$

which are the equations sought, and characterize every straight line that can be drawn through the point (x', y', z') .

If the given point be the origin, then $x' = 0, y' = 0, z' = 0$, and the equations of a line passing through it are therefore

$$x = az, y = bz.$$

PROBLEM III.

(217.) To find the equations of a straight line which passes through two given points.

Let the two given points be (x', y', z') , and (x'', y'', z'') , then the equations of the line passing through one of the points (x'', y'', z'') are

$$\left. \begin{aligned} x - x'' &= a(z - z'') \\ y - y'' &= b(z - z'') \end{aligned} \right\} \dots (1),$$

and, in order that this line may pass also through the other point (x', y', z') , there must exist the conditions

$$x' - x'' = a(z' - z'') \text{ and } y' - y'' = b(z' - z''),$$

which determine for a and b the values

$$a = \frac{x' - x''}{x' - x''}, b = \frac{y' - y''}{x' - x''}$$

These values of a and b being substituted in equations (1), or in equations (2), last problem, either of which characterizes a line through one of the points, will furnish the equations sought, which are, therefore, indifferently either

$$\left. \begin{aligned} x - x'' &= \frac{x' - x''}{x' - x''} (x - x'') \\ y - y'' &= \frac{y' - y''}{x' - x''} (x - x'') \end{aligned} \right\}$$

or

$$\left. \begin{aligned} x - x' &= \frac{x' - x''}{x' - x''} (x - x') \\ y - y' &= \frac{y' - y''}{x' - x''} (x - x') \end{aligned} \right\}.$$

If one of the points (x'', y'', z'') be the origin, then the first pair equations become

$$x = \frac{x'}{x} z, y = \frac{y'}{x} z,$$

which remain the same, whether the other point be (x', y', z'), or ($-x', -y', -z'$) so that, if (x', y', z') be a point on a straight line passing through the origin, ($-x', -y', -z'$) will also be a point on the line.

PROBLEM IV.

(218.) To find the equations of the straight line which passes through a given point, and is parallel to a given line.

Let (x', y', z') be the given point, and let the equations of the given straight line be

$$\left. \begin{aligned} x &= a'z + \alpha' \\ y &= b'z + \beta' \end{aligned} \right\}.$$

Then the equations of any line passing through the given point are

$$\left. \begin{aligned} x - x' &= a(z - z') \\ y - y' &= b(z - z') \end{aligned} \right\}$$

and, in order that this line may be parallel to the former, its projections on the vertical planes must be parallel to the projections of the former line; in other words, they must cut the axis of z at the same angles, so that we must have $a = a'$, $b = b'$; therefore the equations required are

$$\left. \begin{aligned} x - x' &= a'(z - z') \\ y - y' &= b'(z - z') \end{aligned} \right\}.$$

PROBLEM V.

(219.) To determine the conditions requisite for the intersection of two straight lines in space, and to find the coordinates of the point of intersection.

If two straight lines of which the equations are

$$\left. \begin{aligned} x &= az + \alpha \\ y &= bz + \beta \end{aligned} \right\} \text{ and } \left\{ \begin{aligned} x &= a'z + \alpha' \\ y &= b'z + \beta' \end{aligned} \right.$$

intersect, the coordinates of the point of intersection will be the same for both lines; hence, in order to discover what relation must exist among the constants in these equations in this case, we must

eliminate the variables; and we obtain, first by subtraction,

$$(a - a')z + a - a' = 0$$

$$(b - b')z + \beta - \beta' = 0,$$

and then by division,

$$z = \frac{a' - a}{a - a'} = \frac{\beta' - \beta}{b - b'}$$

hence the relation among the constants, when the lines intersect, is fixed by the equation

$$(a' - a)(b - b') = (\beta' - \beta)(a - a').$$

For the coordinates of the point of intersection we have, by substituting the value of z , just deduced, in the expressions for x and y ,

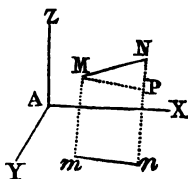
$$x = \frac{aa' - a'\alpha}{a - a'}, y = \frac{b\beta' - b'\beta}{b - b'}, z = \frac{a' - a}{a - a'}.$$

If $a = a'$, and $b = b'$, these expressions for the coordinates of the point of intersection become infinite; therefore, this point being infinitely distant, the proposed lines must be parallel.

PROBLEM VI.

(220.) To find the analytical expression for the distance between two given points in space.

Let M and N be the given points, their coordinates being respectively x', y', z' , and x'', y'', z'' , then, if the points M, N , be projected on the plane of xy , the coordinates x, y , of the projections m, n , will be the same as those of the proposed points; hence for the distance mn we have the expression (14)



$$mn^2 = (x' - x'')^2 + (y' - y'')^2.$$

Now, if MP be drawn parallel to mn , NPM will be a right angle; hence $MN^2 = mn^2 + PN^2 = mn^2 + (Nn - Mm)^2$, that is, calling MN, D, we have

$$D = \sqrt{(x' - x'')^2 + (y' - y'')^2 + (z' - z'')^2}.$$

If one of the points, as (x'', y'', z'') , be the origin, then

$$D = \sqrt{x'^2 + y'^2 + z'^2}.$$

This shows that, in a right-angled paralleliped, the square of the diagonal is equal to the sum of the squares of the three edges.

PROBLEM VII.

(221.) To find the relation that exists among the angles which any straight line makes with the axes of coordinates.

Parallel to any proposed line draw a line from the origin, and let its length, D, represent the radius of the tables, or 1; then if from its extremity parallels be drawn to the three axes terminating in the planes, these parallels will obviously be the cosines of the respective angles which they form with D, or, which is the same thing, they will be the cosines of the angles that D forms with the axes. But the same parallels are the coordinates of the point from which they are drawn; hence we have, by substituting, in the expression for D, (last prob.) $\cos \theta_1$ for x , $\cos \theta_2$ for y , and $\cos \theta_3$ for z , the remarkable relation

$$\cos^2 \theta_1 + \cos^2 \theta_2 + \cos^2 \theta_3 = 1 \dots (1),$$

in which $\theta_1, \theta_2, \theta_3$ denote the angles which any straight line in space makes with the axes of x, y, z .

To determine the value of each cosine, let us suppose that a and b are the tangents of the angles which the projections of the proposed line make with the axis of z ; then the equations of the line D will be $x = az, y = bz$, that is (1)

$$\cos \theta_1 = a \cos \theta_3, \cos \theta_2 = b \cos \theta_3;$$

substituting these values in (1), we obtain the expressions

$$\cos \theta_3 = \frac{1}{\sqrt{a^2 + b^2 + 1}}, \cos \theta_2 = \frac{b}{\sqrt{a^2 + b^2 + 1}}$$

$$\cos \theta_1 = \frac{a}{\sqrt{a^2 + b^2 + 1}}.$$

If (x', y', z') , (x'', y'', z'') be any two points on the proposed line, then (Problem iii.) we have for the tangents of the angles which the line makes with the axis of z , the expressions

$$a = \frac{x' - x''}{z' - z''}, b = \frac{y' - y''}{z' - z''}.$$

Substituting these values in the expressions just deduced, and putting R for the distance between the points, there results

$$\cos \theta_1 = \frac{x' - x''}{R}, \cos \theta_2 = \frac{y' - y''}{R}, \cos \theta_3 = \frac{z' - z''}{R}$$

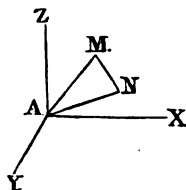
which forms are frequently employed in the doctrine of Forces.

PROBLEM VIII.

(222.) To find the expression for the angle of intersection of two straight lines in space.

Let the two straight lines be represented by the equations

$$\left. \begin{aligned} x &= ax + a \\ y &= bx + \beta \end{aligned} \right\} \text{and} \left\{ \begin{aligned} x &= a'x + a' \\ y &= b'x + \beta' \end{aligned} \right.$$



and, parallel to them, draw, from the origin, the two lines, AM, AN, then we have to determine the angle MAN.

Let AM, AN, each represent the radius, 1, of the tables; then calling the coordinates of M, x', y', z' , and those of N, x'', y'', z'' , we have for the distance, MN, the expression

$$D^2 = (x' - x'')^2 + (y' - y'')^2 + (z' - z'')^2.$$

Now because AM, AN, are each equal to 1, we have, by equation (1), last prob., the conditions

$$x'^2 + y'^2 + z'^2 = 1, \text{ and } x''^2 + y''^2 + z''^2 = 1;$$

therefore, substituting these values in the development of the expression for D^2 , it becomes

$$D^2 = 2 - 2(x'x'' + y'y'' + z'z'').$$

By means of this expression we arrive immediately at that for the cosine of the angle MAN, which we shall call V; for, by trigonometry,

$$\cos V = \frac{AM^2 + AN^2 - MN^2}{2AM \cdot AN} = \frac{2 - D^2}{2}$$

which, by substituting the above value for D^2 , becomes

$$\cos V = x'x'' + y'y'' + z'z'',$$

and this, by last proposition is the same as

$$\cos V = \cos \theta_1 \cos \theta'_1 + \cos \theta_2 \cos \theta'_2 + \cos \theta_3 \cos \theta'_3 \dots (1),$$

where $\theta_1, \theta_2, \theta_3$, denote the angles which AM makes with the axes of x, y, z , and $\theta'_1, \theta'_2, \theta'_3$, denote the angles which AN makes with the same axes.

By substituting for the cosines of these angles their values in terms of a, b , and a', b' , as given in last proposition, the expression (1) takes the form

$$\cos V = \frac{aa' + bb' + 1}{\sqrt{(a^2 + b^2 + 1)(a'^2 + b'^2 + 1)}} \dots (2).$$

If the proposed lines are perpendicular to each other, that is, if $\cos V = 0$, the numerator of this fraction must be 0, that is, we must have the condition

$$aa' + bb' + 1 = 0,$$

or (1)

$$\cos \theta_1 \cos \theta'_1 + \cos \theta_2 \cos \theta'_2 + \cos \theta_3 \cos \theta'_3 = 0.$$

It must be remarked that two straight lines in space may be inclined to each other without intersecting, although this is impossible when both are in the same plane; and the angle of inclination is always measured by that included by two parallels to them, drawn from one point; so that the foregoing expressions for V , the inclination of two straight lines in space, apply, whether they actually intersect or not.

It is also important to observe that the results in the last three problems do not preserve the same form, when the axes of coordinates are oblique; since the expression for D , which enters into these results, becomes obviously more complicated, when the planes are not rectangular. In all the other problems, the inclination of the axes will not affect the form of the results; and the modifications necessary to be introduced into the three problems mentioned, when oblique axes are employed, are given in the supplementary chapter at the end of the volume, which is devoted to "Miscellaneous Propositions in Geometry of three dimensions."

CHAPTER II.

ON THE PLANE.

(223.) If a straight line move in a direction parallel to itself along another straight line, given in position, the surface generated will be a *plane*.

The generating line is sometimes called the *generatrix*, and the line along which it moves the *directrix*.

The intersections of any plane with the coordinate planes are called its *traces*.

PROBLEM I.

(224.) To find the equation of the plane.

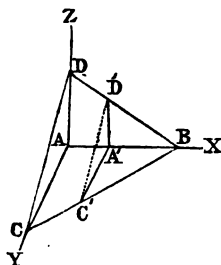
Let BC , BD , DC , be the traces of any proposed plane, which may, therefore, be supposed to be generated by the motion of DC along DB .

Let the equation of the trace BD be

$$z = mx + p \dots (1),$$

and the equation of the trace DC ,

$$z = ny + p \dots (2),$$



p being $= AD$, the z of each trace at the origin, A .

Now, since the generating line is in every position $D'C'$ parallel to DC , the value of z in $D'C'$ will always be

$$z = ny + \beta \dots (3).$$

At the point D' , where this line meets the trace BD , $y = 0$, because this point is in the plane of xz , so that the value of z at the same point is, from equation (3), $z = \beta$. But, by equation (1), the value of z at this point is $z = mx + p$; consequently

$$\beta = mx + p \dots (4),$$

x being the same for every point in $D'C'$ as for D' , because $D'C'$ is throughout equi-distant from the plane of yz . Hence, substituting this expression for β , in equation (3), we shall have the following relation among the coordinates of any point in the proposed plane, viz.

$$z = mx + ny + p \dots (5).$$

This, therefore, is the equation of the plane.

If the coordinate planes are rectangular, as, indeed, we shall always suppose them, m and n will denote the tangents of the angles which the traces BD , DC , make with the axes of x and y respectively. The symbol p denotes the value of z at the origin; if the proposed plane pass through the origin, then $p = 0$, and the equation is

$$z = mx + ny.$$

The equation (5) is usually put under the form

$$Ax + By + Cz + D = 0,$$

being a complete equation of the first degree containing three variables; it comes from equation (5) by multiplying by the arbitrary quantity C , substituting A for mC , B for nC , D for pC , and then transposing all to one side.

PROBLEM II.

(225.) Having given the equation of a plane, to determine the equations of its traces.

Let the equation of the plane be

$$Ax + By + Cz + D = 0,$$

then for every point in this plane, which is situated likewise in the plane of xy , that is, for every point in the trace on the plane of xy , we must have $z = 0$; hence the equation of this trace is

$$Ax + By + D = 0 \dots (1).$$

In like manner, for every point in the trace on the plane of xz , we have $y = 0$; therefore the equation of this trace is

$$Ax + Cz + D = 0 \dots (2).$$

And, similarly, the equation of the trace on the plane of yz is

$$By + Cz + D = 0 \dots (3).$$

If, in (1), we put $y = 0$, the resulting value of x , viz. $x = -\frac{D}{A}$ will be the distance of the origin from the point where the axis of x pierces the proposed plane; or, putting $x = 0$, we have $y = -\frac{D}{B}$ for the distance of the origin from the point where the axis of y pierces the plane. In like manner, for the point where the axis of z pierces the plane, we have $z = -\frac{D}{C}$; hence, when $D = 0$, the plane must pass through the origin.

As to the angles which the traces make with the axes of x, y , we have for their trigonometrical tangents, as given by the three preceding equations, the expressions

$$-\frac{A}{B}, -\frac{A}{C}, -\frac{B}{C}.$$

Let us put p, q, r for the parts of the axes of x, y, z intercepted between the planes and the origin, that is, let us put

$$-\frac{D}{A} = -p, -\frac{D}{B} = -q, -\frac{D}{C} = -r,$$

then the equation of the plane will take this form, viz.

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = -1,$$

in which every term is an abstract number.

PROBLEM III.

(226.) To find the equation of the plane which passes through three given points.

Let the three points be

$$(x', y', z'), (x'', y'', z''), \text{ and } (x''', y''', z'''),$$

then, the form of the equation of the plane being

$$Ax + By + Cz + D = 0,$$

or

$$\frac{A}{D}x + \frac{B}{D}y + \frac{C}{D}z = -1 \dots (1),$$

we have to determine the values of A, B, C, D , so that the following conditions may be fulfilled, viz.

$$\frac{A}{D}x' + \frac{B}{D}y' + \frac{C}{D}z' = -1,$$

$$\frac{A}{D}x'' + \frac{B}{D}y'' + \frac{C}{D}z'' = -1,$$

$$\frac{A}{D}x''' + \frac{B}{D}y''' + \frac{C}{D}z''' = -1.$$

By applying the common operations of algebra (*see Alg. p. 78*), we find for the values of the unknowns, $\frac{A}{D}, \frac{B}{D}, \frac{C}{D}$, the following expressions, viz.

$$\frac{A}{D} = \frac{x'(y'' - y''') - x''(y' - y''') + x'''(y' - y'')}{x'(y''z''' - y'''z'') - x''(y'z''' - y'''z') + x'''(y'z'' - y''z')}$$

$$\frac{B}{D} = \frac{x'(z'' - z''') - x''(z' - z''') + x'''(z' - z'')}{x'(y''z''' - y'''z'') - x''(y'z''' - y'''z') + x'''(y'z'' - y''z')}$$

$$\frac{C}{D} = \frac{y'(x'' - x''') - y''(x' - x''') + y'''(x' - x'')}{x'(y''z''' - y'''z'') - x''(y'z''' - y'''z') + x'''(y'z'' - y''z')}$$

hence the equation of the plane, fulfilling the proposed condition, is determined by substituting these values in equation (1).

If the plane is required to pass through but one point (x', y', z') , then the equation of every such plane is

$$Ax + By + Cz + D = Ax' + By' + Cz' + D,$$

or, rather,

$$A(x - x') + B(y - y') + C(z - z') = 0.$$

PROBLEM IV.

(227.) To determine the conditions which must subsist in order that a straight line may be parallel to a plane.

Let the equation of the plane be

$$Ax + By + Cz + D = 0,$$

and the equations of the straight line

$$\left. \begin{aligned} x &= az + \alpha \\ y &= bz + \beta \end{aligned} \right\}.$$

If these expressions for x and y be substituted in the equation of the plane, the resulting value of z will be that of a point common to both straight line and plane. This value is

$$z = -\frac{A\alpha + B\beta + D}{Aa + Bb + C}$$

which, substituted for z , in the equations of the straight line, give the other two coordinates of the point where the straight line pierces the plane, on the supposition that they are *not* parallel. If the straight line have an indefinite number of points in common with the plane, that is, if it be wholly in the plane, then the foregoing expression for z is susceptible of an indefinite number of values, that is, we must have

$$\frac{A\alpha + B\beta + D}{Aa + Bb + C} = 0$$

so that the *conditions of coincidence* are

$$A\alpha + B\beta + D = 0, \quad Aa + Bb + C = 0 \quad \dots (1).$$

But, if the line is merely parallel to the plane, then, by drawing from the origin a line and plane, respectively parallel to the former, there will be a coincidence; but then $\alpha = 0$, $\beta = 0$, and $D = 0$; hence the conditions (1) become simply

$$Aa + Bb + C = 0 \dots (2),$$

which is the *condition of parallelism*.

Hence, if it be required to draw through a given point (x', y', z') , a straight line parallel to a plane, we have only to substitute, in equations (2), p. 123, any assumed value for one of the coefficients, a, b , and then to determine the other from the condition (2) above.

PROBLEM V.

(228.) To determine the conditions of parallelism of two planes. Let the equations of two planes be

$$Ax + By + Cz + D = 0,$$

$$A'x + B'y + C'z + D' = 0.$$

If they intersect these equations, both exist for the line of intersection; hence, eliminating one of the variables, z , for example, we have, for the other two coordinates of any point in this line, the relation

$$(AC' - A'C)x + (BC' - B'C)y + (DC' - D'C) = 0 \dots (1).$$

But (213) the relation between the coordinates x and y of any straight line in space, is the same as the relation between x and y in the projection of that line on the plane of xy ; consequently equation (1) is that of the projection of the intersection of the two planes on the plane of xy ; and similarly, by eliminating x or y from the proposed equations, we shall obtain the equations of the projection of the same intersection on the plane of yz , or of xz .

When, however, the proposed planes are parallel, the intersection, and, consequently, the projection of it is impossible; so that equation (1) cannot exist for any values of x and y . But so long as the coefficients of x and y , in that equation, exist, the equation itself may be satisfied; for, by giving any arbitrary value to one of the variables, that of the other becomes determinable; so that the equation becomes impossible only when the coefficients of x and y become 0; that is, in order that the planes may be parallel, there must exist the conditions

$$AC' - A'C = 0, \quad BC' - B'C = 0 \dots (2).$$

The same conclusion is immediately derivable from the expressions at (225), for the tangents of the angles which the traces of a plane make with the axes; for, as the traces of two planes on either of the coordinate planes must be parallel, if the planes themselves are, it follows, from the expressions referred to, that we must have

$$\frac{A}{B} = \frac{A'}{B'}, \quad \frac{B}{C} = \frac{B'}{C'}, \quad \frac{A}{C} = \frac{A'}{C'}$$

The first and second of these conditions, which, indeed, include the third, are the same as the conditions (2).

PROBLEM VI.

(229.) A point being given in space, to draw through it a plane parallel to a given plane

Let the equation of the given plane be

$$Ax + By + Cz + D = 0;$$

then, representing the given point by (x', y', z') , the equation of the required plane will take the form (226)

$$A'(x-x') + B'(y-y') + C'(z-z') = 0.$$

But, since the two planes are parallel, we have, by the conditions of parallelism,

$$A' = \frac{A}{C} C', \quad B' = \frac{B}{C} C';$$

hence the equation becomes, by substitution,

$$A(x-x') + B(y-y') + C(z-z') = 0,$$

or

$$Ax + By + Cz - (Ax' + By' + Cz') = 0,$$

or finally,

$$Ax + By + Cz + D' = 0,$$

where D' is put for $-(Ax' + By' + Cz')$; so that, if two planes are parallel, it is always possible to render the first three coefficients in their equations the same in each. If the point (x', y', z') is the origin, then $D' = 0$, and the equation is

$$Ax + By + Cz = 0,$$

which characterizes every plane passing through the origin.

PROBLEM VII.

(230.) To determine the conditions which must subsist, in order that a straight line may be perpendicular to a plane.

Let the equations of the projections of the straight line be

$$x = \alpha z + a \dots (1),$$

$$y = \beta z + \beta \dots (2),$$

and the equation of the plane

$$Ax + By + Cz + D = 0.$$

Then, since the line is perpendicular to the plane, every plane passing through the line must be also perpendicular to the same plane; hence the planes which project the line will each be perpendicular both to the proposed plane and to the coordinate plane on which the projection is made. But a plane which is perpendicular to two planes is perpendicular to their intersection; hence the projecting planes are perpendicular to the traces of the proposed plane; but if a plane is perpendicular to a line, every line in that plane is perpendicular to the same line; and, as the projections of any line are in the projecting planes, it follows, therefore, that if these latter are perpendicular to any traces, so also are the projections. Now, for the traces of the proposed plane, we have the equations

$$\left. \begin{array}{l} Ax + Cz + D = 0 \\ By + Cz + D = 0 \end{array} \right\} \text{ or } \begin{cases} x = -\frac{C}{A}z - \frac{D}{A} \\ y = -\frac{C}{B}z - \frac{D}{B} \end{cases}$$

and we have to express that the lines represented by these equations are respectively perpendicular to those represented by equations (1), (2). This is done by putting (11)

$$a = \frac{A}{C} \text{ and } b = \frac{B}{C}$$

which are the conditions required.

PROBLEM VIII.

(231.) To draw a perpendicular from a given point to a plane, and to determine its length.

Let the plane be represented by the equation

$$Ax + By + Cz + D = 0 \dots (1),$$

then, if the given point be (x', y', z') , the equations of the required line will take the form

$$\left. \begin{aligned} x - x' &= a(z - z') \\ y - y' &= b(z - z') \end{aligned} \right\}$$

But, for this line to be perpendicular to the plane, we must have, by last problem,

$$a = \frac{A}{C}, \quad b = \frac{B}{C}$$

hence the equations of the proposed line are

$$\left. \begin{aligned} x - x' &= \frac{A}{C}(z - z') \\ y - y' &= \frac{B}{C}(z - z') \end{aligned} \right\} \dots (2).$$

If we knew the point where this perpendicular meets the plane, we could at once determine its length, from the expression at (220) for the distance between two given points. Now, since at this unknown point the coordinates are the same both for the perpendicular and the plane, we shall determine it by finding what values of x , y , and z will satisfy the equations (1) and (2); when it will remain only to substitute these values in the expression

$$P = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} \dots (3).$$

As, however, the unknowns x , y , z are seen to enter these three equations, each connected by the same sign, minus, with another quantity, it will be better, instead of seeking to separate them, to aim at performing the elimination without disturbing the connection. We shall endeavour, therefore, to find the values of the expressions $x - x'$, $y - y'$, $z - z'$.

For this purpose, assume

$$Ax' + By' + Cz' + D = D',$$

therefore, substituting for a' , b' , the values

$$a' = \frac{A}{C}, \quad b' = \frac{B}{C}$$

and denoting the inclination of the proposed line and plane by I , we have

$$\sin I = \frac{Aa + Bb + C}{(a^2 + b^2 + c^2)(A^2 + B^2 + C^2)}$$

If the line is parallel to the plane, $\sin I = 0$; therefore, as before determined, (227), the condition of parallelism is

$$Aa + Bb + C = 0.$$

PROBLEM X.

(233.) To determine the inclination of two given planes.

Let the equations of the given planes be

$$Ax + By + Cz + D = 0 \dots (1),$$

$$A'x + B'y + C'z + D' = 0 \dots (2);$$

then, if to each plane a perpendicular line be drawn, the inclination of these perpendiculars will be the inclination of the planes; hence representing the perpendiculars by the equations

$$\left. \begin{aligned} x &= ax + \alpha \\ y &= bx + \beta \end{aligned} \right\} \quad \left. \begin{aligned} x &= a'x + \alpha' \\ y &= b'x + \alpha' \end{aligned} \right\}$$

we must have the relations

$$a = \frac{A}{C}, \quad b = \frac{B}{C}, \quad a' = \frac{A'}{C'}, \quad b' = \frac{B'}{C'}$$

and, therefore, for the angle, V , of inclination sought, we have

$$\begin{aligned}\cos V &= \frac{aa' + bb' + 1}{\sqrt{(a^2 + b^2 + 1)(a'^2 + b'^2 + 1)}} \\ &= \frac{AA' + BB' + CC'}{(A^2 + B^2 + C^2)(A'^2 + B'^2 + C'^2)}.\end{aligned}$$

If the two planes are parallel, then $\cos V = 1$, and we have the condition

$$(AA' + BB' + CC')^2 = (A^2 + B^2 + C^2)(A'^2 + B'^2 + C'^2),$$

which reduces to

$$(AB' - BA')^2 + (AC' - CA')^2 + (BC' - CB')^2 = 0.$$

But the square of any quantity being always positive, the sum of any number of squares can never be 0, unless they themselves are 0; hence the final conditions are

$$AB' - BA' = 0, \quad AC' - CA' = 0, \quad BC' - CB' = 0,$$

as before determined (228).

If the planes are perpendicular to each other, then $\cos V = 0$, so that, in this case, we have the condition

$$AA' + BB' + CC' = 0.$$

If one of the planes, the second, for instance, coincide with one of the coordinate planes, as the plane of xy , then, in equation (2), $z = 0$ and x and y may be any values whatever; consequently that equation cannot subsist, unless

$$A' = 0, \quad B' = 0, \quad \text{and} \quad D' = 0;$$

hence, substituting these values in the expression for $\cos V$, we have

$$\cos V' = \frac{C}{\sqrt{A^2 + B^2 + C^2}}$$

the inclination of the plane (1) to the plane of xz .

In like manner,

$$\cos V'' = \frac{B}{\sqrt{A^2 + B^2 + C^2}}$$

the inclination to the plane of xz .

And

$$\cos V''' = \frac{A}{\sqrt{A^2 + B^2 + C^2}}$$

the inclination to the plane of yz .

By adding the squares of these three last equations together, we obtain the condition

$$\cos^2 V' + \cos^2 V'' + \cos^2 V''' = 1,$$

so that the relation (221), of the inclinations of a line to the coordinate axes, is analogous to that of the inclinations of a plane to the coordinate planes.

The expressions for the inclinations of the second plane (2) to the coordinate planes will be obtained by accenting the letters A , B , C , in those just deduced; and if we represent them by $\cos U'$, $\cos U''$, and $\cos U'''$, the expression for $\cos V$ will be the same as

$$\cos V = \cos V'U' + \cos V'' \cos U'' + \cos V''' \cos U''',$$

which result is also analogous to that at (222).

When the angle V is right, we must have the condition

$$\cos V' \cos U' + \cos V'' \cos U'' + \cos V''' \cos U''' = 0.$$

Scholium.

(234.) The results of the last four problems become much more complicated, when oblique coordinates are employed, instead of rectangular; but of the preceding problems in this chapter the results preserve the same form, whether the coordinates are rectangular or not; and the changes which become necessary in the other cases are considered in the supplementary chapter, at the end.

Before we conclude the present chapter, it will be necessary to prove the converse of the inference in prop. 1., viz. that every equation of the first degree containing three variables, such as

$$Ax + By + Cz + D = 0$$

is the analytical representation of some plane.

For, putting this equation under the form

$$z = -\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C},$$

and then constructing, on two vertical planes, perpendicular to each other, and which may be designated as the planes of xz and of yz , two lines, of which the equations are

$$z = -\frac{A}{C}x - \frac{D}{C}$$

and

$$z = -\frac{B}{C}y - \frac{D}{C};$$

the lines thus constructed may be regarded as the traces of some plane; hence, finding, by prob. 1., the plane of which these are the traces, we fall upon the equation

$$z = -\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C}$$

or

$$Ax + By + Cz + D = 0.$$

We may further remark, that several of the results of the foregoing problems were affected with the ambiguous sign \pm , although the *geometrical* conditions of the problem in each case implied a single unambiguous value. But we have not thought it necessary to enter into any lengthened explanation of these apparent discrepancies between the geometrical and the analytical results of the same problem; deeming our observations at page 42, in Part I., in reference to the like occurrences in geometry, of two dimensions, to be amply sufficient to satisfy any scruple as to the perfect compatibility of *both* results with the premises from which they have respectively been deduced. We therefore refer the student to those observations, merely adding here that, whatever amount of indeterminateness may belong to any analytical result, the same amount of indeterminateness will always, upon examination, be found to attach to the original conditions which have led to it; a position which applies with equal truth to every process of correct reasoning, since the conclusion, in its fullest extent, is necessarily implied in the premises, the development of which is the only office that the reasoning performs.

SECTION III.

ON SURFACES OF THE SECOND ORDER.

(235.) A SURFACE is said to be of the *second order*, when it may be analytically represented by an equation of the second degree, containing three variables.

CHAPTER I.

ON THE SPHERE, AND ON CYLINDRICAL AND CONICAL SURFACES.

PROBLEM I.

(236.) To determine the equation of the sphere.

Let r represent the radius of a sphere, and α, β, γ , the coordinates of its centre; let, also, (x, y, z) denote any point on the surface of the sphere.

Then, r being the distance between the points (α, β, γ) and (x, y, z) , we have (220).

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = r^2,$$

or, by developing,

$$x^2 + y^2 + z^2 - 2\alpha x - 2\beta y - 2\gamma z + \alpha^2 + \beta^2 + \gamma^2 = r^2,$$

the general equation of the sphere, when related to rectangular axes.

If the origin is on the surface of the sphere, then $\alpha^2 + \beta^2 + \gamma^2 = r^2$, and, therefore, the equation becomes

$$x^2 + y^2 + z^2 - 2\alpha x - 2\beta y - 2\gamma z = 0.$$

If the origin is at the centre, then, the coordinates of the centre being each 0, the equation is

$$x^2 + y^2 + z^2 = r^2.$$

If one of the coordinate planes, as the plane of xy , passes through the centre, then $\gamma = 0$, and the equation is

$$(x - \alpha)^2 + (y - \beta)^2 + z^2 = r^2;$$

and, if one of the axes, as the axis of x , pass through the centre, then $\beta = 0$, and $\gamma = 0$, and the equation is

$$(x - \alpha)^2 + y^2 + z^2 = r^2.$$

PROBLEM II.

(237.) To determine the intersection of a sphere with a plane.

Let p represent the distance of the intersecting plane from the centre of the sphere, and constitute three coordinate planes, originating at the centre, one of which, as the plane of xy , may be parallel to the cutting plane. Then every point in the intersecting plane will be given by the equation $z = p$; and, consequently, by the equation in last problem, all the points on the surface of the sphere, which are also common to the plane, must be given by the

equation

$$x^2 + y^2 = r^2 - p^2,$$

which represents a circle; this is, therefore, the intersection.

PROBLEM III.

(238.) To determine the equation of the tangent plane passing through a given point on the surface of the sphere.

Let the given point be (x', y', z') , then the equation of a plane passing through it is

$$A(x - x') + B(y - y') + C(z - z') = 0 \dots (1),$$

and, since the same point is on the surface of a sphere, we must have

$$(x' - \alpha)^2 + (y' - \beta)^2 + (z' - \gamma)^2 = r^2 \dots (2).$$

Now, for the radius of this sphere, that is, for the line passing through the points (α, β, γ) , and (x', y', z') , we have (217) the equations

$$\left. \begin{aligned} x - x' &= \frac{x' - \alpha}{z' - \gamma} (z - z') \\ y - y' &= \frac{y' - \beta}{z' - \gamma} (z - z') \end{aligned} \right\}$$

and it remains to express that this line is perpendicular to the plane (1). By (230) the conditions of perpendicularity are

$$A = \alpha C, \quad B = \beta C,$$

that is,

$$A = \frac{x' - \alpha}{z' - \gamma} C, \quad B = \frac{y' - \beta}{z' - \gamma} C.$$

Hence, substituting these values in (1), and dividing by C, we have for the tangent plane sought the equation

$$(x' - \alpha)(x - x') + (y' - \beta)(y - y') + (z' - \gamma)(z - z') = 0 \dots (3).$$

As this equation must exist in conjunction with (2), which is the same as

$$(x' - \alpha)(x' - \alpha) + (y' - \beta)(y' - \beta) + (z' - \gamma)(z' - \gamma) = r^2,$$

we have, by adding the two together,

$$(x' - \alpha)(x - \alpha) + (y' - \beta)(y - \beta) + (z' - \gamma)(z - \gamma) = r^2,$$

which alone characterizes a plane touching a sphere, of which the radius is r , and the centre (α, β, γ) , in the point (x', y', z') .

If the origin is at the centre, then $\alpha = 0$, $\beta = 0$, and $\gamma = 0$, and the equation of the tangent plane is

$$x'x + y'y + z'z = r^2.$$

Cylindrical Surfaces.

(239.) The name *cylindrical surface* is given to every surface which can be generated by a straight line moving parallel to itself, and, at the same time, describing with its extremity a curve line.

The curve described by the extremity of the generating line is called the *directrix*; and, when it is a plane curve, is usually supposed, for simplicity, to be situated in one of the coordinate planes, the plane of xy .

PROBLEM IV.

(240.) To determine the general equation of a cylindrical surface.
Let the equations of the generating line in any position be

$$\left. \begin{aligned} x &= az + \alpha \\ y &= bz + \beta \end{aligned} \right\} \therefore \left\{ \begin{aligned} \alpha &= x - az \\ \beta &= y - bz \end{aligned} \right.$$

then, since the line is always parallel to itself, the values of a and b will remain the same for any other position; but α, β , representing the x, y , of the point where the line meets the plane of xy , necessarily vary with this point. Now, the point is always in the directrix: and, as this is a given curve, the relation between x and y , that is, between α and β , is given. Hence, if in the equation expressing the relation between x and y , for every point in the directrix, that is, if in the equation of the directrix we substitute for x and y the foregoing values of α and β , the result will be the equation of the surface sought.

Thus, if the equation of the directrix be represented by the function

$$F(x, y) = 0,$$

that of the cylindrical surface will be

$$F(x - az, y - bz) = 0.$$

For example, if it be required to find the equation of an oblique cylinder of circular base, then, supposing the base to be in the plane of xy , and the origin of the axes to be at the centre, the equation of the base, or of the directrix, will be

$$x^2 + y^2 = r^2;$$

therefore, substituting in this, $x - az$ for x , and $y - bz$ for y , we have, for the equation sought,

$$(x - az)^2 + (y - bz)^2 = r^2.$$

If the base had been an ellipse, characterized by the equation

$$A^2y^2 + B^2x^2 = A^2B^2,$$

then the equation of the cylinder would have been

$$A^2(y - bz)^2 + B^2(x - az)^2 = A^2B^2.$$

If the cylinder is right instead of oblique, then $a = 0$, and $b = 0$, and the equation of the cylinder becomes then the same as the equation of the directrix; observing, however, that the equation of the directrix is always supposed to be accompanied by the condition $z = 0$, because the curve is considered as wholly in the plane of xy . But no such condition accompanies the equation of the right cylinder; on the contrary, z may be taken of any value whatever: so that, while the equations

$$x^2 + y^2 = r^2, \quad z = 0,$$

represent a circle on the plane of xy , the equations

$$x^2 + y^2 = r^2, \quad z = \frac{0}{0}$$

represent the right cylinder, having that circle for its base.

Conical Surfaces.

(241.) A *conical surface* is that generated by a straight line which constantly passes through the same point in space, and describes with its extremity a curve line.

The given point is called the *vertex*, or, sometimes, the *centre* of the conical surface; and the curve line, described by the generating line, is the *directrix*, which, when a plane curve, is usually supposed to be situated in the plane of xy .

From its mode of generation, it is obvious that a conical surface consists of two portions, united to each other by the vertex, which is the only point common to each. These two portions are called *sheets*, and thus the conical surface is said to be composed of two sheets.

PROBLEM V.

(242.) To determine the general equation of a conical surface.

Let (x', y', z') represent the vertex, or centre of the surface; then, since the generating line always passes through this point, its equations in any position will be (216)

$$\left. \begin{aligned} x - x' &= a(z - z') \\ y - y' &= b(z - z') \end{aligned} \right\} \text{ or } \begin{cases} x = az + (x' - az') \\ y = bz + (y' - bz') \end{cases}$$

$$\therefore a = \frac{x - x'}{z - z'}, \quad b = \frac{y - y'}{z - z'}$$

Now here, as in the preceding problem, $(x' - az')$, $(y' - bz')$, is the x, y , of the point where the generating line pierces the plane of xy , which point is, therefore, always in the directrix; but the x, y of

every such point is given by the equation of the directrix; hence, substituting, in the equation of the directrix, the values $(x' - ax')$, and $(y' - by')$, for x and y , we shall obtain the equation of the surface.

Thus, if the equation of the directrix be represented by the function

$$F(x, y) = 0,$$

that of the conical surface will be

$$F(x' - ax', y' - by') = 0,$$

where a and b involve the variable coordinates. By substituting for a and b their values above, the equation is

$$F\left(\frac{x'z - xz'}{z - z'}, \frac{y'z - yz'}{z - z'}\right) = 0.$$

Let it be required to find the equation of an oblique cone of circular base.

Suppose the base to be situated in the plane of xy , and the axes to originate at its centre; then the equation of the base, or of the directrix, is

$$x^2 + y^2 = r^2.$$

Substituting, in this equation,

$$\frac{x'z - xz'}{z - z'} \quad \text{and} \quad \frac{y'z - yz'}{z - z'}$$

for x and y , we have, for the equation sought,

$$(x'z - xz')^2 + (y'z - yz')^2 = r^2(z - z')^2.$$

If the cone is *right*, that is, if the axis of the cone coincides with the axis of z , then $x' = 0$, $y' = 0$, and this equation becomes

$$z'^2x^2 + z'^2y^2 = r^2(z - z')^2.$$

(243.) Let it now be required to find the equation of a right cone, having an elliptical base.

Assuming, as before, the centre of the base for the origin, the equation of the directrix will be

$$A^2y^2 + B^2x^2 = A^2B^2,$$

and as the vertex is in the axis of z , we have $x' = 0$, and $y' = 0$; hence, instead of x and y , in the equation of the directrix, we must substitute

$$-\frac{zs'}{z-z'}, \quad \text{and} \quad -\frac{ys'}{z-z'};$$

and we thus have for the equation of the surface

$$A^2y^2 + B^2x^2 = \left(\frac{z-z'}{z'}\right)^2 A^2B^2.$$

If we put Z for $z - z'$, m for $\frac{A}{z'}$, and n for $\frac{B}{z'}$, the equation of the elliptic right cone takes this simple form,

$$m^2y^2 + n^2x^2 = m^2n^2Z^2.$$

(244.) Lastly, let it be required to find the equation of an oblique circular cone when the origin of the axes is not at the centre.

Here the equation of the directrix is

$$(x - \alpha)^2 + (y - \beta)^2 = r^2,$$

in which, if we substitute for x and y the values

$$\frac{x'z - \alpha z'}{z - z'}, \quad \frac{y'z - \beta z'}{z - z'}$$

we have for the equation of the surface,

$$[x'z - xz' - a(z - z')]^2 + [y'z - yz' - \beta(z - z')]^2 = r^2(z - z')^2.$$

If the origin of the axes be on the circumference of the base, and one of them, as the axis of x , pass through the centre, then $a = r$, and $\beta = 0$, and the equation of the directrix is

$$x^2 + y^2 = 2rx;$$

therefore the equation of the conical surface is

$$(x'z - xz')^2 + (y'z - yz')^2 = 2r(z - z')(x'z - xz').$$

CHAPTER II.

ON SURFACES OF REVOLUTION.

(245.) EVERY curve surface generated by the revolution of a curve round a fixed axis is called a *surface of revolution*.

Hence the characteristic of surfaces of revolution is this, viz. that every section made by a plane perpendicular to the fixed axis is a circle, whose centre is in that axis.

The equations to the several surfaces of revolution will take the most simple and commodious form, by supposing the fixed axis (sometimes called the axis of revolution) to coincide with one of the coordinate axes, as the axis of z . We shall here suppose this coincidence of the axis of revolution with the axis of z .

PROBLEM I.

(246.) To determine the general equation of a surface generated by the revolution of a plane curve about an axis in that plane.

Let DC be any position of the generating curve; then, drawing DM parallel to AZ , and joining AM , we shall have AM , MD , equal to the coordinates of the point D , as given by the equation of the plane curve, DC . Put $AM = r$, and $MD = z$; then, since in the equation of the curve one of the variables must be a function, F , of the other, we have

$$z = F(r).$$

Now r is always equal to the radius of the circle described by the point D round the axis AZ , and it is therefore related to the x and y of that point by the equation

$$r = \sqrt{x^2 + y^2};$$

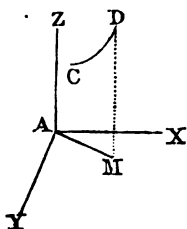
hence the relation of the coordinates x , y , z , of every point, D , in the surface of revolution, is given by the equation

$$z = F(x^2 + y^2)^{\frac{1}{2}};$$

this, then, is the general equation of a surface of revolution, under the condition that the revolving curve is always in the plane of the fixed axis.

(247.) As a first example, let it be required to find the equation of the surface generated by the revolution of any straight line about the axis of z , and in the same plane with it.

Here the equation of the generating line in any position, when referred to the axis of z as axis of ordinates, and a perpendicular to it from the origin, as axis of abscissas, is



$$z = F(r) = ar + b,$$

therefore, substituting $\sqrt{x^2 + y^2}$ for r , we have for the equation of the surface,

$$(z - b)^2 = a^2(x^2 + y^2).$$

Now, b is the ordinate, z' , of the generating line at the origin; and the tangent a is the same as $\frac{z'}{r}$;* hence, by substitution, the equation becomes

$$(z - z')^2 r^2 = z'^2 x^2 + z'^2 y^2,$$

which agrees with the equation at Prob. v. in last Chapter, as it ought; for the surface here considered is obviously that of a right cone.

(248.) Let it now be required to find the equation of the surface described by the revolution of an ellipse about one of its axes.

1. Let the minor axis be the fixed axis, coinciding with the axis of z , then the equation of the generating curve will be

$$A^2 z^2 + B^2 r^2 = A^2 B^2;$$

hence, substituting $x^2 + y^2$ for r^2 , we have

$$A^2 z^2 + B^2 (x^2 + y^2) = A^2 B^2,$$

the equation of the *ellipsoid of revolution*.

2. If the revolution be about the major axis, then the generating curve is represented by the equation

* r being here the abscissa of the point where the line cuts the axis of abscissas, or the radius of the circle described by the generatrix on the plane of xy .

$$B^2z^2 + A^2r^2 = A^2B^2,$$

and the surface by the equation

$$B^2z^2 + A^2(x^2 + y^2) = A^2B^2.$$

If $A = B$, we have

$$z^2 + x^2 + y^2 = A^2$$

for the equation of a sphere, as before found.

The ellipsoid of revolution is generally called a *spheroid*; a *prolate spheroid* when the revolution is about the minor axis; and an *oblate spheroid* when the revolution is about the major axis.

(249.) Let the surface be described by the revolution of an hyperbola about one of its axes.

Suppose, first, that the second or conjugate axis of the hyperbola is that which is fixed; then the equation of the generating curve is

$$A^2z^2 - B^2r^2 = -A^2B^2,$$

and, putting $x^2 + y^2$ for r^2 , it becomes

$$B^2(x^2 + y^2) - A^2z^2 = A^2B^2,$$

the equation of the *hyperboloid of revolution of a single sheet*.

Suppose, secondly, that the revolution is about the transverse axis, then it is obvious that the surface generated will consist of two sheets. The equation of the generating curve will be

$$B^2z^2 - A^2r^2 = A^2B^2,$$

and that of the surface,

$$B^2z^2 - A^2(x^2 + y^2) = A^2B^2,$$

the equation of the *hyperboloid of revolution of two sheets*

If the asymptotes revolve with the curve, then in each of the preceding cases there will be generated a conical surface of two sheets, asymptotic to the hyperboloid.

(250.) Let now the generating curve be a parabola revolving about its principal diameter; then the equation of the generating curve is

$$r^2 = pz;$$

and, consequently, for the surface, we have

$$x^2 + y^2 = pz,$$

the equation of the *paraboloid of revolution*.*

Lastly, suppose the generating curve to be a circle, in the same plane as the axis of z , and having its centre in the plane of xy .

The equation of this circle is

$$(r - a)^2 + z^2 = R^2,$$

and, consequently, that of the surface generated by it is

$$\{\sqrt{x^2 + y^2} - a\}^2 + z^2 = R^2;$$

which equation, when cleared of radicals, is of the fourth order. From the mode of its generation, it is easy to see that it is a ring-like or *annular surface*; and that its meridian sections, or those formed by planes passing through the fixed axis, are all circles. This annular surface is called the *torus*.

* If the curve revolve about the other axis, the surface generated will be of the fourth order.

CHAPTER III.

ON SURFACES OF THE SECOND ORDER IN GENERAL.

THE equations which have just been shown to characterize surfaces of revolution are obviously only so many particular forms of the more general equations

$$Lx^2 + My^2 + Nz^2 = P \dots (1),$$

$$Lx^2 + My^2 = Qz \dots (2).$$

The first of these comprehends the ellipsoid and hyperboloid of revolution, and the second contains the paraboloid of revolution.

We here propose, by discussing these equations, to ascertain in general the nature and characteristics of the two classes of surfaces to which the preceding belong, leaving it to be shown in the next chapter that these two classes, together with the cylinder, comprehend all the surfaces of the second order.

(251.) Before we proceed to the discussion of the equations (1) and (2), we shall remark that the surfaces which they represent naturally divide themselves into two distinct classes; those which have a centre and those which have not. The former class are represented by equation (1) and the latter by equation (2). This may be readily shown; thus, let (x', y', z') , be any point on a surface, represented by equation (1), and from this point let there be drawn a straight line through the origin, then we know (217) that $(-x', -y', -z')$ will be also a point on this line; but the same

is likewise a point on the surface, for the equation (1) remains unaltered, whether the coordinates x, y, z , be positive or negative. Now these points are at the same distance from the origin, viz. $D = \sqrt{x^2 + y^2 + z^2}$, therefore every straight line drawn through the origin, and terminating in the surface, is bisected at the origin, which point is, therefore, the centre of the surface.

Equation (2) cannot represent any surface which has a centre; for if it could, the origin of the rectangular axes might be removed to that centre. If the z of this supposed centre be c , then the z of any point in the surface would be $z + c$; so that the equation (2), when thus transformed, would still have a term containing only the first power of z ; hence, if through this new origin a straight line from any point (x', y', z') , in the surface be drawn, the point $(-x', -y', -z')$, in the same line, equally distant from the origin, cannot belong to the surface, for we cannot change z into $-z$, in the equation of the surface, without producing a change in the sign of the term involving z . Hence the surface has no centre.

(252.) If equation (1) be solved for x , we find two values numerically equal, but of contrary signs; so that the plane of yz divides into two equal parts every chord drawn parallel to the axis of x . What has been said of the plane of yz equally applies to the planes of xz , and of xy ; viz. each of these bisects all the chords drawn parallel to the intersection of the other two. Any three planes, each possessing this property of bisecting every chord drawn parallel to the intersection of the other two, are called a system of *diametral planes*. Those which we have just noticed are no other than the rectangular coordinate planes; they are distinguished as the *principal diametral planes*.* The curves traced on these planes, by their intersections with the surface, are called the *principal sections*; and the intersections of the same planes, that is, the axes of coordinates, are called the *principal axes of the surface*.

* It will be hereafter proved that there can be but one system of rectangular diametral planes.

(253.) As to the surfaces represented by equation (2), we find, by proceeding as above, that they have but two principal diametral planes, viz. the planes of xy and of yz ; nevertheless the traces of the surface on the three coordinate planes are called the principal sections, and the coordinate axes the principal axes of the surface.

On Surfaces which have a Centre.

(254.) We shall now proceed to the discussion of equation (1), which it will be convenient to consider under each of the three following forms, viz.

$$Lx^2 + My^2 + Nz^2 = P$$

$$Lx^2 + My^2 - Nz^2 = P$$

$$Nx^2 - My^2 - Lx^2 = P$$

which forms agree with those that we have already found to characterize the surfaces of revolution which have a centre.

The Ellipsoid.

Let us first take the form

$$Lx^2 + My^2 + Nz^2 = P,$$

which characterizes a surface limited in every direction; for, if any straight line be drawn from the origin, its equations will be

$$x = az, y = bz.$$

If these values of x and y be substituted in the equation of the

surface, we shall have for the z of the point where the line pierces the surface the expression

$$z = \sqrt{\frac{P}{La^2 + Mb^2 + N}}.$$

Now, whatever values be given to a, b , the denominator of this expression can never become 0; hence the value of z , and consequently those of x and y , are real and finite, so that every diameter meets the surface.

To determine the principal sections of the surface we must put successively

$$x = 0, y = 0, z = 0,$$

in the proposed equation, and we shall thus have

$$My^2 + Nz^2 = P, \text{ the trace on the plane of } yz,$$

$$Lx^2 + Nz^2 = P \quad xz,$$

$$Lx^2 + My^2 = P \quad xy.$$

These equations characterize ellipses referred to their principal diameters; these diameters therefore coincide with the principal axes of the surface. If $P = 0$, each ellipse will be reduced to a point, viz. the origin of the axes. By supposing P negative, the sections become imaginary, showing that in this case no surface exists.

Let us now examine the sections parallel to these principal sections, and made by planes, whose respective distances from the principal planes may be represented by

$$x = \pm \alpha, y = \pm \beta, z = \pm \gamma.$$

The equations of these sections will be

$$La^2 + My^2 + Nz^2 = P, \text{ section parallel to } yz,$$

$$Lx^2 + M\beta^2 + Nz^2 = P \quad xz,$$

$$Lx^2 + My^2 + N\gamma^2 = P \quad xy.$$

These equations also represent ellipses referred to their principal diameters; hence their centres must be on the axes of coordinates. All the parallel sections are moreover *similar ellipses*, since their axes, which are found by putting first one and then the other variable equal to zero, are in a constant ratio whatever be the distance of the section from the principal section.

But, in order that these ellipses may exist, the quantities

$$P - L\alpha^2, P - M\beta^2, P - N\gamma^2$$

must be positive; for, if such values, positive or negative, be given to α , β , and γ , as to render these expressions 0, then each ellipse is reduced to a point; and if greater values than these be given to α , β , and γ , the sections become impossible. Hence the surface is entirely comprised within six tangent planes, drawn parallel to the coordinate planes, and of which the distances from the origin, or centre of the surface, are

$$A = \sqrt{\frac{P}{L}}, B = \sqrt{\frac{P}{M}}, C = \sqrt{\frac{P}{N}}$$

A, B, C, being put for the distances of the centre from the planes which limit the surface in the directions of x , y , z , respectively; in other words, A, B, C, represent the principal semi-axes of the surface,* which, from the nature of the several sections, is called the *ellipsoid*.

From the foregoing expressions for the principal semi-axes, we get

$$L = \frac{P}{A^2}, M = \frac{P}{B^2}, N = \frac{P}{C^2}$$

* The principal semi-diameters are also immediately obtained from the proposed equation of the surface; thus, at the point where the axis of x pierces the surface, y and z are 0; hence for this point the equation gives

$$x = A = \sqrt{\frac{P}{L}}. \text{ In like manner, } y = B = \sqrt{\frac{P}{M}}, \text{ and } z = C = \sqrt{\frac{P}{N}}.$$

hence the equation of the surface becomes, by substitution,

$$A^2 B^2 z^2 + A^2 C^2 y^2 + B^2 C^2 x^2 = A^2 B^2 C^2,$$

or

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1,$$

the equation of the ellipsoid, related to its principal axes.

If any two of the semi-axes, A, B, C , be equal, then also two of the coefficients, L, M, N , will be equal; and hence one system of parallel sections must be circles, and therefore the surface will be an ellipsoid of revolution. Thus, if $B = C$, then $M = N$, and we have

$$A^2(z^2 + y^2) + B^2 x^2 = A^2 B^2, \text{ or } \frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{B^2} = 1$$

for the equation of the *ellipsoid of revolution* about the axis of x .

If $A = B = C$, the surface is spherical, having the equation

$$x^2 + y^2 + z^2 = A^2.$$

Hence the varieties of the ellipsoid are the *ellipsoid of revolution*, the *sphere*, a *point*, and an *imaginary surface*.

The Hyperboloid of a single Sheet.

(255.) The second form of the general equation of central surfaces of the second order is

$$Lx^2 + My^2 - Nz^2 = P.$$

The traces, or principal sections of the surface here represented, are

$My^2 - Nz^2 = P$, the trace on the plane of yz ,

$$Lx^2 - Nz^2 = P \quad xz,$$

$$Lx^2 + My^2 = P \quad xy.$$

Of these sections the first two are hyperbolas, and the third an ellipse; and, as each curve is referred by its equation to its principal diameters, these diameters coincide with the principal axes of the surface. If $P = 0$, the hyperbolas each degenerate into a system of two intersecting straight lines, and the ellipse reduces to a point. The surface in this case will be an elliptic cone, as will be seen presently.

For the sections parallel to the principal sections, made by planes whose distances from the origin are

$$x = \pm \alpha, y = \pm \beta, z = \pm \gamma,$$

we have the equations

$$My^2 - Nz^2 = P - L\alpha^2, \text{ section parallel to } yz.$$

$$Lx^2 - Nz^2 = P - M\beta^2 \quad xz,$$

$$Lx^2 + My^2 = P + N\gamma^2 \quad xy.$$

Here, as before, the two former sections are hyperbolas, and the last an ellipse, which enlarges as γ increases; the smallest being that in the plane of xy , answering to $\gamma = 0$, and called by the French writers the *ellipse de gorge* or *throat* of the surface.

Each series of parallel sections consists of *similar* figures, since their axes are to each other always in the same ratio; and they are obviously always possible, however distant the intersecting planes may be from the origin; so that this surface is unlimited in every direction; it does not, however, meet the axis of z , for putting both x and $y = 0$, in its equation, the resulting value of z is imaginary, viz.

$$z = \sqrt{-\frac{P}{N}}.$$

But, for the other principal semi-axes of the surface, the same equation gives

$$x = \sqrt{\frac{P}{L}}, y = \sqrt{\frac{P}{M}}.$$

Calling these latter A, B, and the former C $\sqrt{-1}$, and introducing these terms into the equation of the surface, which, because of its continuity, and from the nature of its sections, is called the hyperboloid of a single sheet, we have

$$A^2 B^2 z^2 - A^2 C^2 y^2 - B^2 C^2 x^2 = -A^2 B^2 C^2,$$

or

$$\frac{z^2}{A^2} + \frac{y^2}{B^2} - \frac{x^2}{C^2} = 1,$$

the equation of the hyperboloid of one sheet related to its principal axes.

It has already been seen, that, when $P = 0$, in the proposed equation, the trace of the surface on the plane of xy is merely a point; but every section parallel to this plane on either side of it is an ellipse, given by the equation

$$Lx^2 + My^2 = N\gamma^2.$$

By making successively $x = 0$, and $y = 0$, we find, for the semi-axes of the ellipse, the values

$$y = \gamma \sqrt{\frac{N}{M}}, x = \gamma \sqrt{\frac{N}{L}}$$

these increase with γ , that is, the elliptic sections increase as the intersecting plane recedes from the plane of xy ; hence the surface can be no other than the elliptic right cone, having its vertex at the origin.

This conical surface is asymptotic to the hyperboloid, as may be thus proved.

Let the hyperboloid and cone be both cut by a plane, parallel to the plane of xy , at the distance, γ , from the origin; then the equations of the two elliptic sections will be respectively,

$$Lx^2 + My^2 = P + N\gamma^2 \quad \text{and} \quad Lx^2 + My^2 = N\gamma^2.$$

Now, if these sections have any point in common, the coordinates, x, y , of that point will be the same in each; so that we must have

$$P + N\gamma^2 = N\gamma^2 \therefore \frac{P}{N\gamma^2} + 1 = 1,$$

which is impossible, unless γ be infinite.

If $A = B$, in the equation of the hyperboloid, then $L = M$, and the sections parallel to the plane of xy become circles; hence the equation

$$C^2(x^2 + y^2) - A^2z^2 = A^2C^2, \quad \text{or} \quad \frac{x^2}{A^2} + \frac{y^2}{A^2} - \frac{z^2}{C^2} = 1$$

characterizes the hyperboloid of revolution of a single sheet about the axis of z .

Hence the varieties of the hyperboloid of one sheet are the hyperboloid of revolution, and the conical surface.

It appears from the above that the equation of the elliptic right cone, having its vertex at the origin, is

$$Lx^2 + My^2 = Nz^2,$$

or, substituting for the coefficients of this equation their values

$$L = \frac{P}{A^2}, \quad M = \frac{P}{B^2}, \quad N = \frac{P}{C^2}$$

and dividing by P , it becomes

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = \frac{z^2}{C^2}$$

or

$$A^2C^2y^2 + B^2C^2x^2 = A^2B^2z^2,$$

in which equation, A, B, represent the semi-axes of the elliptic section, which is at the distance, C, from the vertex.

If we put m for $\frac{A}{C}$, and n for $\frac{B}{C}$, the equation becomes

$$m^2 y^2 + n^2 x^2 = m^2 n^2 z^2,$$

agreeing with the form at (243).

Or, if we put p for $\frac{C}{A}$, and q for $\frac{C}{B}$, the form is

$$qy^2 + px^2 = z^2;$$

hence, if it were required to express the equation of an elliptic right cone, of which the section, two feet from the vertex, has for principal semi-diameters the lengths 5 feet and 7 feet; then, since $A = 7$, $B = 5$, and $C = 2$, the equation is

$$\frac{4}{49}y^2 + \frac{4}{25}x^2 = z^2.$$

The Hyperboloid of two Sheets.

(256.) The third species of central surfaces is represented by the equation

$$Nx^2 - My^2 - Lz^2 = P.$$

For the principal sections we have the equations

$$Nx^2 - My^2 = P, \text{ the trace on the plane of } yz,$$

$$Nx^2 - Lz^2 = P \quad xz,$$

$$My^2 + Lz^2 = -P \quad xy.$$

Of these sections the first two are hyperbolas, referred to their principal diameters, which, therefore, coincide with the principal

axes of the surface. The second section being imaginary, shows that the surface does not meet the plane of xy ; hence of the former hyperbolic sections one branch of each hyperbola is situated above, and the other below, the plane of xy ; so that the surface consists of *two sheets*. If $P = 0$, the hyperbolas degenerate into a system of straight lines, intersecting at the origin, and the trace on xy is a point.

The sections parallel to the traces, and of which the distances from the origin are respectively

$$x = \pm \alpha, y = \pm \beta, z = \pm \gamma,$$

are given by the equations

$$Nz^2 - My^2 = P + L\alpha^2, \text{ section parallel to } yz,$$

$$Nx^2 - Lx^2 = P + M\beta^2 \quad xz,$$

$$My^2 + Lx^2 = N\gamma^2 - P \quad xy.$$

Hence the sections parallel to the planes of yz , and xz , are all hyperbolas, having their centres on the axis of x and y ; and each figure is similar to the principal section to which its plane is parallel. The sections parallel to the plane of xy are similar ellipses, provided the distance $\pm \gamma$, of the cutting plane from the origin, be not so small as to render $N\gamma^2 - P$ negative; for, if it be, the plane will not meet the surface. Beyond these limits, however, the section is always real, and that for $+\gamma$ always equal to that for $-\gamma$; the sheets are, therefore, infinite, and are symmetrically situated with respect to the plane of xy . If

$$N\gamma^2 - P = 0, \text{ then } +\gamma = \sqrt{\frac{P}{N}}, \text{ and } -\gamma = -\sqrt{\frac{P}{N}}$$

therefore, at these distances, each section reduces to a point, and no part of the surface can be between these tangent planes; so that this value of γ is the vertical semi-axis of the surface; the two horizontal axes are imaginary. The expressions for them, given

by putting in the proposed equation, first $z = 0$, $y = 0$, and then $x = 0$, $z = 0$, are

$$x = \sqrt{-\frac{P}{L}}, y = \sqrt{-\frac{P}{M}}$$

Putting $A\sqrt{-1}$ and $B\sqrt{-1}$ for these semi-axes, and C for the former one, and then, as before, introducing these expressions into the original equation, we have

$$A^2 B^2 z^2 - A^2 C^2 y^2 - B^2 C^2 x^2 = A^2 B^2 C^2,$$

or

$$\frac{z^2}{A^2} + \frac{y^2}{B^2} - \frac{x^2}{C^2} = -1$$

the equation of the hyperboloid of two sheets related to its principal axes.

When $P = 0$, in the equation proposed, it may be proved, as in art. (255), that the equation then represents an elliptical conic surface asymptotic to the hyperboloid.

If $A = B$, then $L = M$, and the sections parallel to the plane of xy , become circles; hence the equation

$$A^2 z^2 - C^2 (x^2 + y^2) = A^2 C^2, \text{ or } \frac{x^2}{A^2} + \frac{y^2}{A^2} - \frac{z^2}{C^2} = -1$$

represents the hyperboloid of revolution of two sheets. Hence this and the elliptic right cone are the varieties of the surface.

On Surfaces which have not a Centre.

(257.) We now proceed to examine the second class of surfaces, and which are represented by the equation

$$Lx^2 + My^2 = Qz.$$

This equation involves in it the forms

$$Lx^2 + My^2 = Qz, \text{ and } Lx^2 - My^2 = Qz, \text{ or } My^2 - Lx^2 = Qz.$$

The last two equations represent the same surfaces, the axes of x and y being merely interchanged.

The Elliptic Paraboloid.

By putting, in the equation

$$Lx^2 + My^2 = Qz,$$

the successive values $x = 0, y = 0, z = 0$,

we have, for the principal sections of the surface, the equations

$$My^2 = Qz, \text{ the trace on the plane of } yz,$$

$$Lx^2 = Qz, \quad \quad \quad xz,$$

$$Lx^2 + My^2 = 0 \quad \quad \quad xy.$$

Hence the traces on the planes of yz and xz are parabolas, referred to their principal axes, which, therefore, coincide with the axes of the surface. Of these traces the principal diameter of the first coincides with the axis of z , and the second diameter with the axis of y ; the principal diameter of the second trace coincides with the axis of z , and the second diameter with the axis of x .

The third trace, or that on the plane of xy , is merely a point, viz. the origin of the axes; this plane therefore *touches* the surface.

For the sections parallel to the traces, and whose distances from the origin are respectively

$$x = \pm \alpha, \quad y = \pm \beta, \quad z = \gamma,$$

we have the equations

$$My^2 = Qz - Lx^2, \text{ section parallel to } yz,$$

$$Lx^2 = Qz - M\beta^2 \quad xz,$$

$$Lx^2 + My^2 = N\gamma \quad xy.$$

The first two sections are, like their parallel traces, parabolas: they are not only *like*, but are also *equal*, to the parallel traces; for the y, z of every parabolic section parallel to yz , is measured, not from the *vertex* of the parabola, but from a point $x = \pm \alpha$, on the axis of x . That the z, y may be measured from the vertex we must

put $z + \frac{Lx^2}{Q}$ for z in the equation of the section, which substitution transforms it to $My^2 = Qz$; thus the parabola is equal to the trace on yz . A like transformation shows that the trace on xz is equal to every section parallel to it. The third section is an ellipse, whose centre is on the axis of x , and principal diameters parallel to the axes of x and y ; and it will always be possible, however great the distance, γ , above the plane of xy may be; so that the surface has no limit above this plane:—below it the sections are impossible. These elliptic sections are all *similar*; since the semi-axes, found by putting successively $x = 0$ and $y = 0$, in the last equation, have the same ratio whatever be γ .

From the nature of its sections, this surface is called the *elliptic paraboloid*. It may be generated by the parabola in the plane of yz moving parallel to itself, with its vertex always in contact with the trace in the plane of xz ; or by the motion of this latter trace, under like restrictions, along the former kept fixed. The concavities of both parabolas are obviously towards the *same* parts.

If $L = M$, the elliptic sections become circular, and then the surface is one of revolution about the axis of x ; hence the equation

$$x^2 + y^2 = px,$$

where $p = \frac{Q}{L}$, represents the *elliptic paraboloid of revolution*, which is the only variety of the elliptic paraboloid.

The Hyperbolic Paraboloid.

(258.) The surface represented by the equation

$$Lx^2 - My^2 = Qz$$

has for its traces the equations

$$My^2 = -Qz, \text{ the trace on the plane of } yz,$$

$$Lx^2 = Qz, \quad xz,$$

$$Lx^2 - My^2 = 0, \quad xy.$$

The first two traces are parabolas, and the principal diameter of each coincides with the axis of z . The origin is the common vertex of both parabolas; but that in the plane of yz is *below*, and that in the plane of xz is *above*, the plane of xy . The trace on the plane of xy is merely a system of two straight lines intersecting at the origin, their equations being $x = \pm y \sqrt{\frac{M}{L}}$.

For the sections made by the planes

$$z = \pm \alpha, \quad y = \pm \beta, \quad x = \pm \gamma,$$

we have the equations

$$My^2 = L\alpha^2 - Qz, \text{ section parallel to } yz,$$

$$Lx^2 = M\beta^2 + Qz, \quad xz,$$

$$Lx^2 - My^2 = \pm Q\gamma \quad xy.$$

The first two sections are parabolas, whose vertices are *not* on the axes, but the principal diameter of each is parallel to the axis of z ; that of the first parabola is, however, in the direction of z negative, and that of the second in the direction of z positive.

But let us examine these sections more narrowly. And, first, it may be remarked that the coordinates, y, z , of every point in the section parallel to yz are measured from the point $x = \pm a$ of the axis of x ; in other words, this point is the origin of the coordinates, y, z , of the section. Let us then remove this origin to the vertex of the parabola: this is done by substituting $z + \frac{Lx^2}{Q}$ for z in the

equation, which then becomes $My^2 = -Qz$; this equation being the same as that of the trace on the plane of yz , it follows that all the sections parallel to this plane are equal parabolas.

Similar remarks apply to the sections parallel to the plane of xz ; but in these the vertices of the parabolas are all below the plane of xy , while, in the former, the vertices are above that plane, as the transformation shows. Moreover, the vertices of the former series are all in the plane of xz , and are therefore upon the trace in that plane:—those of the latter are all in the trace upon the plane of yz .

The third section, or that parallel to the plane of xy , is an hyperbola related to its principal diameters; its centre is, therefore, on the axis of z . The form of its equation, however, shows that when the section is above the plane of xy , the transverse diameter is parallel to the axis of x , and the conjugate parallel to the axis of y ; but, when the section is below the plane of xy , then, on the contrary, the transverse axis is parallel to the axis of y , and the conjugate to the axis of x ; and at equal distances, γ and $-\gamma$, above and below the plane of xy , the transverse axis of the one section is the same absolute length as the conjugate axis of the other, and vice versa. The sections above the plane are similar figures, and those below are also similar, as is evident from their equations. It is also obvious from what has just been said, that if any equi-distant sections be projected upon the plane of xy , they will furnish a pair of *conjugate hyperbolas* on that plane.

Of all the hyperbolic sections above the plane of xy the vertices are situated on the parabolic trace, on the plane of xz ; for, putting $y = 0$, in the equation

$$Lx^2 - My^2 = Q\gamma.$$

we have, for the corresponding value of x , the semi-transverse axis of the hyperbola, or, which is the same thing, the distance of the vertex from the axis of z , the expression

$$x^2 = \frac{Q}{L} \gamma.$$

But, at the same distance, $z = \gamma$, from the plane of xy , there is a point in the parabolic trace referred to, of which the distance from the axis of z is given by the same expression, viz.

$$x^2 = \frac{Q}{L} \gamma;$$

hence these two points coincide.

In a similar manner, it may be shown that of all the hyperbolic sections below the plane of xy the vertices are situated on the parabolic trace on the plane of yz .

Having thus seen that all the sections parallel to the plane of yz are parabolas, that these parabolas are all equal, and that their vertices are all on the parabolic trace on the plane of xz , it follows that if the parabola in the plane of yz be moved parallel to itself, its vertex always being in contact with the parabola in the plane of xz , the surface we are now considering will be generated. It will be also generated by keeping fixed the parabola which we have here supposed to move, and moving the other under like restrictions. The concavities of the parabolas are evidently towards *opposite* parts.

From the nature of its sections, this surface is called the *hyperbolic paraboloid*; and, as no modification of the coefficients can ever render one of these sections a *closed* curve, the surface, it would seem, can never be one of revolution; a fact fully established at page 210.

The equation of the asymptotes of any hyperbolic section

$$Lx^2 - My^2 = \pm Qz,$$

is known to be (page 141, Part I.),

$$Lx^2 - My^2 = 0, \text{ or } x = \pm y \sqrt{\frac{M}{L}}.$$

Hence, if these two lines are constructed on the plane of xy , the perpendicular planes passing through them will be those which contain the asymptotes of all the hyperbolic sections. Now these lines thus constructed on the plane of xy , are the very lines into which the hyperbolic section degenerates, when the cutting plane coincides with the plane of xy , as we have already seen by the equations of the traces; therefore planes drawn through these, perpendicular to the plane of xy , continually approach, but never meet, the surface, except at their intersections with the plane of xy .

The plane of xy , although having two straight lines in it common to the surface, is, nevertheless, the *tangent plane* at the origin, as we shall presently see (pa. 185); so that, in all the non-central surfaces, one of the rectangular coordinate planes (that of xy ,) is *tangent* to the surface.

From articles (257) and (258) it appears that the traces on the planes of yz and xz , of these surfaces, are parabolas, whose parameters are

$$p = \pm \frac{Q}{M}, \text{ and } p' = \frac{Q}{L}$$

$$\therefore M = \pm \frac{Q}{p}, \text{ and } L = \frac{Q}{p'}.$$

Hence, substituting these values in the general equation

$$Lx^2 + My^2 = Qz,$$

of these surfaces, and dividing by Q , we have

$$\frac{x^2}{p'} \pm \frac{y^2}{p} = z,$$

or

$$px^2 \pm p'y^2 = pp'z,$$

for the equation of the paraboloid, which is elliptic, or hyperbolic, according as the upper or lower sign of p' has place.

Tangent Planes to Surfaces of the Second Order.

(259.) If a straight line meet a surface in but one point, it is said to be a *linear tangent* to the surface at that point; and, since an indefinite number of straight lines may be drawn through a given point, there is obviously no limit to the number of linear tangents at that point. The surface, which is the locus of these tangents, is called the *tangent plane* at the proposed point, and that it is a plane we shall presently see.

PROBLEM.

To find the equation of a tangent plane, drawn through any point on a central surface of the second order.

Let x', y', z' , be the coordinates of the proposed point on any surface, included in the general equation.

$$Lx^2 + My^2 + Nz^2 = P \dots \dots (1).$$

Then any linear secant passing through the same point will be represented by the equations

$$\left. \begin{aligned} x - x' &= a (z - z') \\ y - y' &= b (z - z') \end{aligned} \right\} \dots \dots (2),$$

in which a and b are quite arbitrary, because the secant may take any direction whatever.

From equation (1) subtract

$$Lx'^2 + My'^2 + Nz'^2 = P$$

and there results

$$L(x^2 - x'^2) + M(y^2 - y'^2) + N(z^2 - z'^2) = 0,$$

or

$$L \frac{x - x'}{z - z'} (x + x') + M \frac{y - y'}{z - z'} (y + y') + N (z + z') = 0.$$

When each secant becomes a tangent, then, at the point (x', y', z') of contact, $x = x'$, $y = y'$, and $z = z'$, and the various values of a and b , or, which is the same thing, of $\frac{x - x'}{z - z'}$, and $\frac{y - y'}{z - z'}$, no longer remain arbitrary in equation (2), or in the equation just deduced, but become subject to the relation

$$L \frac{x - x'}{z - z'} z' + M \frac{y - y'}{z - z'} y' + Nz' = 0 \dots \dots (3).$$

Now this equation remains the same for every point (x, y, z) , in each tangent, and not exclusively for the point of contact with the surface, as is manifest from equations (2), which show that the values

$$\frac{x - x'}{z - z'}, \frac{y - y'}{z - z'}$$

are constant for every point (x, y, z) on the line represented by

those equations. Hence equation (3) represents the surface in which all the lineal tangents through (x', y', z') are situated; the surface is therefore a plane. By developing this equation, it takes the form

$$Lx'z + My'y + Nz'z = Lx'^2 + My'^2 + Nz'^2 = P,$$

or

$$\frac{x'x}{A^2} + \frac{y'y}{B^2} + \frac{z'z}{C^2} = 1,$$

which is therefore, the *equation of the tangent plane*.

It appears, from equation (3), that, if a straight line (2) touch the surface, the relation between the constants, a, b , must be such as to satisfy the equation

$$La x' + Mb y' + Nz' = 0.$$

The equations to the *normal*, or straight line drawn from the point of contact perpendicular to the tangent plane, are

$$x - x' = \frac{Lx'}{Nz'} (z - z')$$

$$y - y' = \frac{My'}{Nz'} (z - z').$$

If $M = N$, then the surface is of revolution about the axis of x ; and the last of these equations, which represents the projection of the normal on the plane of xy , reduces to

$$x'y = y'x, \text{ or } y = \frac{y'}{x'} x;$$

hence this projection passes through the origin, and, consequently, the normal must cut the axis of x .

In like manner it may be shown that if $L = N$, that is, if the surface revolve about the axis of y , the normal must cut that axis; and the projection on the plane of xy will in a similar way show that, when $L = M$, the normal will cut the axis of x , about which the surface revolves.

PROBLEM.

To find the equation of the tangent plane when the surface has not a centre.

By page 181, the surfaces which have not a centre may be represented by the equation

$$px^2 \pm p'y^2 = pp'z \dots \dots (1).$$

Any linear secant drawn through a point (x', y', z') on this surface will be represented, as before, by the equations

$$\left. \begin{aligned} z - z' &= a(z - z') \\ y - y' &= b(z - z') \end{aligned} \right\} \dots \dots (2);$$

in which a and b , varying with the direction of the secant, may take any values whatever.

From equation (1) take

$$px^2 \pm p'y^2 = pp'z,$$

and we have

$$p(z^2 - z'^2) \pm p'(y^2 - y'^2) = pp'(z - z'),$$

or

$$p \frac{z - z'}{z + z'} (z + z') \pm p' \frac{y - y'}{y + y'} (y + y') = pp'.$$

Now, when each secant becomes a tangent, then at the point (x', y', z') of contact, we have $x = x'$, $y = y'$, and $z = z'$; and, therefore, the coefficients a, b , in equation (2), become subject to the condition

$$2pax' \pm 2p'by' = pp' \dots (3),$$

for the values

$$\frac{x - x'}{z - z'}, \quad \frac{y - y'}{z - z'}$$

remain the same for every point in the same secant, or in the same tangent; hence, substituting these values for a, b , in (3), we have, for the equation of the surface in which all the linear tangents are situated,

$$2p \frac{x - x'}{z - z'} x' \pm 2p' \frac{y - y'}{z - z'} y' = pp';$$

this represents a plane:—when developed, it becomes

$$\begin{aligned} 2pxx' \pm 2p'yy' &= 2pp'z' + pp'(z - z') \\ &= pp'(z + z'), \end{aligned}$$

the equation of the tangent plane.

At the origin, where x, y, z are each 0, this equation becomes $z = 0$; hence the plane of xy is tangent to the surface.

Equation (3) expresses the conditions which the constants a, b , must have, in order that the straight line (2) may be a linear tangent to the surface.

On Conjugate Diametral Planes.

(260.) We have already defined (252) a system of conjugate diametral planes to be such that each bisects all the chords

drawn parallel to the intersection of the other two; and we have shown that such a system exists in every surface included in the general equation

$$Lx^2 + My^2 + Nz^2 = P \dots\dots (1),$$

provided the coordinate axes are *rectangular*. It will be hereafter shown that there are an infinite number of *oblique* axes to which every such surface may be referred, without altering the form of its equation; and hence we may infer, by imitating the reasoning at (251), that there are an infinite number of systems of oblique diametral planes in central surfaces of the second order, these diametral planes being no other than the oblique coordinate planes.

If, then, we suppose the equation

$$L'x^2 + M'y^2 + N'z^2 = P \dots\dots (2)$$

to represent the same surface as equation (1), when the coordinates are transformed from rectangular to oblique, we may apply to this equation the same reasoning that we have employed in the discussion of equation (1); and the only difference in the results will be that what before were *rectangular* conjugate diameters, will here be *oblique* conjugates, both as regards the surface itself, and the several intersections. From the form of equation (1) it was shown (254, &c.) that a plane drawn through the extremity of a principal diameter (not imaginary), and parallel to the plane of the other two, was necessarily a tangent plane to the surface; hence, since (2) has the same form as (1), we infer that a plane through the extremity of one conjugate diameter, and parallel to the plane of the other two, touches the surface.

From these remarks and from what has been shown in (254), (255), &c., we may infer that, if A' , B' , C' , represent any system of

semi-conjugates belonging to a central surface of the second order, the equation of that surface, referred to them as axes, will be

$$\frac{x^2}{A'^2} + \frac{y^2}{B'^2} + \frac{z^2}{C'^2} = 1,$$

and it will be an ellipsoid, if A'^2, B'^2, C'^2 , are all positive, an hyperboloid of one sheet if only two of these are positive, and an hyperboloid of two sheets if but one is positive.

As to the non-central surfaces, or those comprehended in the general equation

$$px^2 \pm p'y^2 = pp'z,$$

when the surface is referred to *rectangular* coordinates, it will also be shown of them that there are an infinite number of systems of *oblique* axes in reference to which the equation of the surface will preserve the same form; but as in this class of surfaces there are but *two principal* diametral planes, so are there but *two conjugate* diametral planes in each of these oblique systems. This is obvious from the form of the equation, and from the inference, consequent upon that form, already deduced at page 180.

To prove that for each surface of the second order there exist systems of coordinate axes, in infinite variety, in reference to which the equation of the surface preserves the form under which we have hitherto discussed it, is a problem that we have reserved for the supplementary chapter.

SECTION IV.

CHAPTER I.

ON THE ORTHOGONAL PROJECTIONS OF PLANE SURFACES.

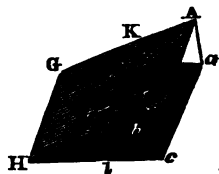
(261.) A PLANE figure is said to be orthogonally projected on a plane when each side of it is projected perpendicularly on that plane.

THEOREM.

(262.) The projection of a plane surface on a plane is equal to the area of that surface multiplied by the cosine of its inclination to the plane of projection.

Since any plane figure may be divided into triangles, it will be sufficient to prove the truth of this theorem in the case of the triangle.

Let, then, ABC be any plane triangle, and let abc be the orthogonal projection of it on any plane, Ha . Produce the plane of ABC to meet the plane of projection in GH ; and perpendicular to AC draw the two parallels, AG , CH ; then, if through B the line KL be drawn parallel to AC , we shall have, for the area of the triangle ABC , the expression



$$\Delta ABC = \frac{1}{2} AC \cdot CL;$$

also, if l be the projection of L , we shall have

$$\Delta abc = \frac{1}{2} AC \cdot cl.$$

Now, if α represent the inclination of the two planes, that is, the angle CHc , we shall have

$$cl = CL \cos \alpha,$$

$$\therefore \Delta abc = \Delta ABC \cdot \cos \alpha.$$

THEOREM II.

(263.) The square of the area of any plane figure is equal to the sum of the squares of its projections on three rectangular planes.*

Let S represent any plane surface, and S' , S'' , S''' , its three projections on the planes of xy , xz , yz ; then, putting α , α' , α'' , for the several inclinations of the plane of S to the coordinate planes, we have, by last theorem,

$$S'^2 = S^2 \cos^2 \alpha, \quad S''^2 = S^2 \cos^2 \alpha', \quad S'''^2 = S^2 \cos^2 \alpha''.$$

Now (pa. 146)

$$\cos^2 \alpha + \cos^2 \alpha' + \cos^2 \alpha'' = 1;$$

hence

$$S^2 = S'^2 + S''^2 + S'''^2.$$

Cor. If the projections of the same surface on three other rectangular planes be S_1' , S_1'' , S_1''' , then, as before,

* By the square of an area is meant the square of its numerical value.

$$S^2 = S_1'^2 + S_1''^2 + S_1'''^2;$$

consequently,

$$S^2 + S'^2 + S''^2 = S_1'^2 + S_1''^2 + S_1'''^2,$$

that is, *the sum of the squares of the projections is the same for every system of rectangular planes.*

THEOREM III.

(264.) If a surface be projected on three rectangular planes, and then these projections be orthogonally projected on a given plane, the sum of these last projections will be equal to the orthogonal projection of the original surface on this given plane.

Let S represent the surface, and S' , S'' , S''' , its projections on the rectangular planes. Let also s' be its projection on any other plane inclined at any angle, V , to the plane of S , then $s' = S \cos V$, also, α , α' , α'' being, as in last problem,

$$S' = S \cos \alpha, \quad S'' = S \cos \alpha', \quad S''' = S \cos \alpha''.$$

Now, if we represent the inclinations of the plane of s' to the rectangular planes, that is, to the planes of S' , S'' , and S''' , by β , β' , β'' , we shall have for $\cos V$ the expression (pa. 146)

$$\cos V = \cos \alpha \cos \beta + \cos \alpha' \cos \beta' + \cos \alpha'' \cos \beta'';$$

and multiplying this by S , we have

$$S \cos V = s' = S' \cos \beta + S'' \cos \beta' + S''' \cos \beta'',$$

which expresses the property announced.

Cor. Also, if S , T , U represent any number of surfaces

situated in different planes, then we have, in a similar manner, for the projections of each of these on a fixed plane, the plane of s , the equations

$$s' = S' \cos \beta + S'' \cos \beta' + S''' \cos \beta'',$$

$$t' = T' \cos \beta + T'' \cos \beta' + T''' \cos \beta'',$$

$$u' = U' \cos \beta + U'' \cos \beta' + U''' \cos \beta'',$$

where T' , T'' , T''' , and U' , U'' , U''' , are the projections of T and U on the rectangular planes; while t' , u' , are the projections of these on the plane of s .

If we represent the sum of the projections on the plane of xy by M' , the sum of those on the plane of xz by M'' , and the sum of those on the plane yz by M''' , while the sum of the projections on the fourth plane is represented by m' , we shall have, by adding together the foregoing equations,

$$m' = M' \cos \beta + M'' \cos \beta' + M''' \cos \beta''.$$

If we introduce a fifth plane, whose inclinations to the rectangular planes are γ , γ' , and γ'' ; and on which the sum of the projections of S , T , U is m'' , we shall have

$$m'' = M' \cos \gamma + M'' \cos \gamma' + M''' \cos \gamma''.$$

In the same way, for a sixth plane,

$$m''' = M' \cos \delta + M'' \cos \delta' + M''' \cos \delta''.$$

Thus the sum of the projections of any series of areas on a plane is equal to the sum of the projections, formed on the same plane, by first projecting all the figures on a system of rectangular planes, and then projecting these projections on the proposed plane.

Cor. 2. If the three planes containing the projections m' , m'' , m''' , are also perpendicular, then, multiplying the foregoing expressions

for these projections by $\cos \beta$, $\cos \gamma$, $\cos \delta$, respectively, and adding the results, we get M' . Similarly, we get M'' and M''' ; hence, keeping in view the relations at page 146, we have

$$M' = m' \cos \beta + m'' \cos \gamma + m''' \cos \delta$$

$$M'' = m' \cos \beta' + m'' \cos \gamma' + m''' \cos \delta'$$

$$M''' = m' \cos \beta'' + m'' \cos \gamma'' + m''' \cos \delta''$$

THEOREM IV.

(265.) The same notation being employed, it is required to prove that

$$m'^2 + m''^2 + m'''^2 = M'^2 + M''^2 + M'''^2,$$

when both systems of planes are rectangular.

By squaring each equation, in the above group, and adding together the results, we have

$$\begin{aligned} M'^2 + M''^2 + M'''^2 &= m'^2 (\cos^2 \beta + \cos^2 \beta' + \cos^2 \beta'') \\ &\quad + m''^2 (\cos^2 \gamma + \cos^2 \gamma' + \cos^2 \gamma'') \\ &\quad + m'''^2 (\cos^2 \delta + \cos^2 \delta' + \cos^2 \delta'') \\ &\quad + 2m'm'' (\cos \beta \cos \gamma + \cos \beta' \cos \gamma' + \cos \beta'' \cos \gamma'') \\ &\quad + 2m'm''' (\cos \beta \cos \delta + \cos \beta' \cos \delta' + \cos \beta'' \cos \delta'') \\ &\quad + 2m''m''' (\cos \gamma \cos \delta + \cos \gamma' \cos \delta' + \cos \gamma'' \cos \delta'') \end{aligned}$$

Now, by art. (233), the factors of m'^2 , m''^2 , and m'''^2 , are each equal to 1, and, by the same art., the factors of $2m'm''$, $2m'm'''$, and $2m''m'''$, are each equal to 0; hence this equation is the same as

$$M'^2 + M''^2 + M'''^2 = m'^2 + m''^2 + m'''^2.$$

This proves that, if any number of plane surfaces, however situated in space, be projected on different systems of rectangular planes, and the projections on each plane be collected into one sum, then the squares of the three sums thus furnished by each system always amount to the same quantity.

Cor. The expression for the sum of the projections on any one of the planes, as the plane of m' , is

$$m' = \sqrt{M'^2 + M''^2 + M'''^2 - m''^2 - m'''^2},$$

which sum will be the greatest possible, when $m'' = 0$, and $m''' = 0$, for then it becomes

$$m' = \sqrt{M'^2 + M''^2 + M'''^2}.$$

Now there is nothing contradictory in supposing $m'' = 0$, and $m''' = 0$; for, since the projection of a surface is equal to that surface multiplied by the cosine of its inclination to the plane of projection, the projection must be considered as positive or negative, according as the cosine is positive or negative, or according as the inclination is acute or obtuse; hence the sum of the projections of any number of surfaces on one of the coordinate planes may become 0, on account of the negative projections equalling the positive; and when this is the case also with another coordinate plane, then, as we have just seen, the projections on the third plane amount to a greater sum than they would do under any other circumstances; this plane is, therefore, called *the plane of greatest projection*.

(266.) We may readily determine the direction of this plane, or its position in reference to the rectangular planes of M' , M'' , M''' , by means of the conditions $m'' = 0$, $m''' = 0$ which exist simultaneously with it; for, introducing these values in the group of

equations originally employed, we have

$$M' = m' \cos \beta, \quad M'' = m' \cos \beta', \quad M''' = m' \cos \beta'';$$

whence

$$\cos \beta = \frac{M'}{m'} = \frac{M'}{\sqrt{M'^2 + M''^2 + M'''^2}}$$

$$\cos \beta' = \frac{M''}{m'} = \frac{M''}{\sqrt{M'^2 + M''^2 + M'''^2}}$$

$$\cos \beta'' = \frac{M'''}{m'} = \frac{M'''}{\sqrt{M'^2 + M''^2 + M'''^2}}$$

in which equations β, β', β'' , denote the inclinations of the plane of greatest projection to the arbitrary rectangular planes of M', M'', M''' , and thus the direction of this plane in reference to any system of rectangular planes is determinable, when the sums M', M'', M''' , of the projections on this system of planes are known.

The situation of the plane of greatest projections is not fixed in space; for the projections on any plane being the same as on any parallel plane, it follows that every plane, having the requisite inclinations to the rectangular planes, possesses the characteristic property of the *principal plane*, or plane of greatest projection.

THEOREM V.

(267.) The sum of the projections on any plane equally inclined to the principal plane is constant.

Let T represent the sum of the projections of any number of surfaces inclined at an angle θ to the principal plane; and let $\epsilon, \epsilon', \epsilon''$, represent its inclinations to the three primitive planes, then (prob. iii.)

$$T = M' \cos \epsilon + M'' \cos \epsilon' + M''' \cos \epsilon'',$$

but (266)

$$M' = m' \cos \beta, \quad M'' = m' \cos \beta', \quad M''' = m' \cos \beta'';$$

hence, by substitution,

$$T = m' (\cos \beta \cos \varepsilon + \cos \beta' \cos \varepsilon' + \cos \beta'' \cos \varepsilon'').$$

Now the expression within the parentheses represents the cosine of the angle θ (233), therefore

$$T = m' \cos \theta,$$

that is,

$$T = \sqrt{M'^2 + M''^2 + M'''^2} \cdot \cos \theta;$$

so that T is constant, if θ is; and if $\cos \theta$ increase or diminish, T will increase or diminish proportionally.

Cor. On every plane perpendicular to the principal plane the sum of the projections is 0; for when $\theta = 90^\circ$, then $T = 0$.

The foregoing theorems on projections find their application in the higher parts of Physical Astronomy.

CHAPTER II.

ON THE TRANSFORMATION OF COORDINATES IN SPACE.

(268.) In order to transform the equation of a surface from one system of axes to another, we must first find expressions for the primitive coordinates in terms of the new; and these, substituted in the original equation, will lead to the transformation desired.

The most simple of these transformations is that in which the new axes are parallel to the old, where the only change is in the position of the origin. In this case, if a, b, c denote the coordinates of the new origin, and (x', y', z') represent any point in the surface, in reference to the old axes, then the coordinates x, y, z , of the same point, in reference to the new, will obviously be

$$x = a + x', \quad y = b + y', \quad z = c + z',$$

which are, therefore, the formulas to be employed, when the origin merely is altered.

If the direction of the axes be altered, then the formulas for substitution are not so readily obtained. We may affirm, however, that the values of the new coordinates must be linear functions of the old; that is, these values must be of the form

$$x = a + mx' + m'y' + m''z'$$

$$y = b + nx' + n'y' + n''z'$$

$$z = c + px' + p'y' + p''z';$$

for these expressions must be such that, when they are substituted in the equation of the plane, the result may not surpass the first degree, which it would, however, do, if either of the above equations surpassed the first degree. Having thus ascertained the form of the required expressions, it remains to determine the constant coefficients $a, m, m', m'',$ &c. If we suppose $x' = 0, y' = 0, z' = 0$, we shall have, for the coordinates of the new origin,

$$x = a, \quad y = b, \quad z = c;$$

these, therefore, are easily determined. The remaining constants must depend on the mutual inclinations of the two systems of axes; and, as these relations will not be disturbed by supposing the two origins to coincide, we shall, for greater simplicity, consider a, b, c , as absent from the foregoing expressions; then x, y, z , representing

the primitive coordinates of any point in space, the new coordinates x', y', z' , of the same point will be related to the former, as in the equations

$$x = mx' + m'y' + m''z'$$

$$y = nx' + n'y' + n''z'$$

$$z = px' + p'y' + p''z'$$

Suppose the point to be situated on the axis of x' ; then $y' = 0$, $z' = 0$, and, consequently,

$$x = mx', \quad y = nx', \quad z = px',$$

$$\therefore m = \frac{x}{x'}, \quad n = \frac{y}{x'}, \quad p = \frac{z}{x'}.$$

(269.) 1. *Let the primitive axes be rectangular, and the new ones oblique.*

Then for any point in the axis of x' , x' will be the hypotenuse, and x the base, of a right-angled triangle; also x' and y will be the hypotenuse and base of a second right-angled triangle; and x', z , will, in like manner, be the hypotenuse and base of a third; hence, calling the angles which the axis of x' makes with the axes of x , of y , and of z , X , Y , and Z , respectively, we have

$$m = \frac{x}{x'} = \cos X, \quad n = \frac{y}{x'} = \cos Y, \quad p = \frac{z}{x'} = \cos Z.$$

For a point situated on the axis of y' , we have, by last article,

$$x = m'y', \quad y = n'y', \quad z = p'y',$$

$$\therefore m' = \frac{x}{y'} = \cos X', \quad n' = \frac{y}{y'} = \cos Y', \quad p' = \frac{z}{y'} = \cos Z',$$

where X', Y', Z' , denote the inclinations of the axis of y' to the axes of x , y , and z , respectively.

In like manner, for any point on the axis of x' , we have

$$x = m''x', \quad y = n''x', \quad z = p''x',$$

$$\therefore m'' = \frac{x}{x'} = \cos X'', \quad n'' = \frac{y}{x'} = \cos Y'', \quad p'' = \frac{z}{x'} = \cos Z'',$$

X'', Y'', Z'' , denoting the inclinations of the axis of x' to the axes of x, y, z , respectively. Hence the expressions for the primitive coordinates in terms of the new are

$$\left. \begin{aligned} x &= x' \cos X + y' \cos X' + z' \cos X'' \\ y &= x' \cos Y + y' \cos Y' + z' \cos Y'' \\ z &= x' \cos Z + y' \cos Z' + z' \cos Z'' \end{aligned} \right\} \dots (A);$$

the nine angles which enter into these expressions being subject to the conditions (221)

$$\left. \begin{aligned} \cos^2 X + \cos^2 Y + \cos^2 Z &= 1 \\ \cos^2 X' + \cos^2 Y' + \cos^2 Z' &= 1 \\ \cos^2 X'' + \cos^2 Y'' + \cos^2 Z'' &= 1 \end{aligned} \right\} \dots (1),$$

the values, in other respects, being arbitrary. But if the angles which the new axes form among themselves are given or fixed, then six of the foregoing angles become dependent on the other three, which are arbitrary; for, let

$$V = \text{the angle } [x', y'],$$

$$U = \text{the angle } [y', z'],$$

$$W = \text{the angle } [z', x'],$$

then (222) besides the preceding conditions, we must also have

$$\left. \begin{aligned} \cos V &= \cos X \cos X' + \cos Y \cos Y' + \cos Z \cos Z' \\ \cos U &= \cos X \cos X'' + \cos Y \cos Y'' + \cos Z \cos Z'' \\ \cos W &= \cos X' \cos X'' + \cos Y' \cos Y'' + \cos Z' \cos Z'' \end{aligned} \right\} \dots (2);$$

we have thus altogether six equations, which are insufficient to determine the nine angles involved in them; but, by giving any arbitrary values not inconsistent with the conditions (1) to three of these, the remaining six are all deducible from the given equations. Hence, that we may be able to fix the positions of the new axes in space, we must know the angles which they make with each other, and three of the angles which they make with the primitive system. If we know only the angles which the new system of axes make with each other; then this system may take any position whatever, in reference to the primitive system.

(270.) 2. *Let both systems be rectangular.*

In this case

$$\cos V = 0, \quad \cos U = 0, \quad \cos W = 0;$$

hence the equations (2) become

$$\left. \begin{aligned} \cos X \cos X' + \cos Y \cos Y' + \cos Z \cos Z' &= 0 \\ \cos X \cos X'' + \cos Y \cos Y'' + \cos Z \cos Z'' &= 0 \\ \cos X' \cos X'' + \cos Y' \cos Y'' + \cos Z' \cos Z'' &= 0 \end{aligned} \right\} \dots (3),$$

so that three of the angles formed by the new system with the primitive being determined, the remaining six are given by the equations (1) and (3), and thus the constants, that is, the several cosines, in (A), the formula of transformation, become known.

If one of the new axes, as the axis of x' , coincide with the primitive axis of x , in case 1, and V be the angle formed by the other two; and if both of these happen to be situated in the plane of xy , the formulas of transformation become very simple; for since, in this case,

$\cos U = 0$, $\cos W = 0$, $\cos X'' = 0$, $\cos Y'' = 0$, and $\cos Z'' = 1$,
equations (2) give

$$\cos Z = 0, \quad \cos Z' = 0;$$

hence equations (1) reduce to

$$\left. \begin{aligned} \cos^2 X + \cos^2 Y &= 1 \\ \cos^2 X' + \cos^2 Y' &= 1 \end{aligned} \right\} \therefore \begin{cases} \cos Y = \sin X \\ \cos Y' = \sin X' \end{cases}$$

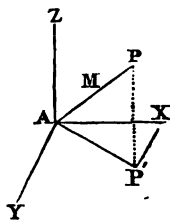
and the formulas (A) are, in this case,

$$x = x' \cos X + y' \cos X', \quad y = x' \sin X + y' \sin X' \dots (A'),$$

which are the same as those already given at page 85, Part I., to pass from a system of rectangular axes to any other system of axes situated in the same plane.

(271.) *To pass from rectangular to polar coordinates.*

Let P be any point (x, y, z) in space, and draw AP from the origin; then A may be considered as the pole, and $AP = r$ the radius vector of the point P. Let AM be made equal to unity, or the radius of the tables; then, denoting the inclinations of AP to the axes of x, y, z , by α, β, γ , we shall have (221) for the coordinates of M, the values $\cos \alpha, \cos \beta, \cos \gamma$; consequently, for the coordinates of P, we have the values



$$x = r \cos \alpha, \quad y = r \cos \beta, \quad z = r \cos \gamma \dots (B),$$

which must exist in conjunction with the condition

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

Other formulas for transforming rectangular to polar coordinates may be obtained, in which only two angles enter, viz. the angle PAP' , formed by the radius vector and its projection on the plane of xy ; and the angle $P'AX$, formed by this projection and the axis of x . Call the first of these angles θ , and the second ϕ ; then, if r' represent the projection of r , the right-angled triangles $PP'A$, $P'XA$, give

$$r' = r \cos \theta, \quad x = r' \cos \phi, \quad y = r' \sin \phi, \quad z = r \sin \theta;$$

hence

$$x = r \cos \theta \cos \phi, \quad y = r \cos \theta \sin \phi, \quad z = r \sin \theta \dots (C).$$

Note. When the new origin does not coincide with the primitive, then the coordinates a, b, c of the new origin must be added to the expressions for x, y, z in the preceding formulas.

With regard to the signs of the trigonometrical quantities, which enter the formulas (C), we must observe that, by supposing ϕ to vary from 0 to 360° , as in (129), and θ to vary from 0 to $\pm 90^\circ$, while the sign of r always remains positive, the formulas will in all cases correctly mark out the position of the point whose coordinates they express. Hence the radius vector ought always to be considered positive. In like manner the position of the point (x, y, z) is correctly determined by the signs of the cosines which enter the formulas (B), r being considered positive.

(272.) Of the preceding formulas of transformation, those which enable us to pass from one system of rectangular axes to another are the most important. To effect this transformation, the knowledge of three angles only is requisite, as we have already seen (270); but, as nine angles enter the formulas (A), the remaining six must be determined from the equations of condition (1) and (3). To remedy this inconvenience, new formulas were given by Euler, and afterwards employed by Lagrange and Laplace, which dispensed with the equations of condition; and gave at once the

Let now the great circle TS be drawn; then, in the spherical triangle TSR, we have

ST = angle formed by the axis of x' and axis of y ,

TR = ψ , SR = $90^\circ - \phi$, and $\angle R = 180^\circ - \theta$,

$$\therefore \cos ST = \cos Y = \sin \phi \cos \psi - \cos \phi \sin \psi \cos \theta.$$

From the point where AZ pierces the sphere draw a great circle Z'P, through T, then ZTP = 90° , then $\angle P = 90^\circ$; hence the spherical triangle TPR gives

TP = complement of the angle formed by the axes of x' and z ,

$$\therefore \sin TP = \cos Z = \sin \theta \sin \psi;$$

we have thus obtained expressions for three of the cosines which enter the formulas (A), viz. for $\cos X$, $\cos Y$, and $\cos Z$.

Again, let U be the point where the axis of y' pierces the sphere, and complete the spherical triangles URQ,* URS, in the first of which we have

UQ = angle formed by the axis of y' and axis of x .

UR = UT + TR = $90^\circ + \psi$, RQ = ϕ , and $\angle R = \theta$,

$$\therefore \cos UQ = \cos X' = \cos \theta \cos \psi \cos \phi - \sin \psi \cos \phi,$$

and in the second triangle, URS, we have

US = angle formed by the axis of y' and axis of y ,

RS = $90^\circ - \phi$, UR = $90^\circ + \psi$, and $\angle R = \theta$,

$$\therefore \cos US = \cos Y' = -\cos \psi \cos \phi \cos \theta - \sin \psi \sin \phi.$$

* RT prolonged will pass through U, because AR, AX', AY', are in one plane.

Drawing, now, the quadrantal arc ZUL, the triangle URL, right-angled at L, gives

UL = complement of the angle formed by the axes of y' and z ,

$$UR = 90^\circ + \psi, \text{ and } \angle R = \theta,$$

$$\therefore \sin UL = \cos Z' = \cos \psi \sin \theta,$$

we have thus expressions for three more of the angles which enter (A), and there still remain three to determine. Let V be the point where the axis of x' pierces the sphere, and draw the arcs VQ and VSK, the former meeting TR in N, and the latter meeting the production of TR in K; then AZ' being perpendicular to the plane of the circle TNRK, N and K will be right angles. The triangle NQR gives

NQ = VQ - VN = — complement of the inclination of the axes
of x' and x ,

$$RQ = \phi, \quad \angle R = \theta,$$

$$\therefore \sin NQ = -\cos X' = \sin \phi \sin \theta.$$

The triangle KSR gives

SK = VK - VS = — complement of the inclinations of the axes
of x' and y ,

$$SR = SQ - QR = \text{complement of } \phi, \quad \angle R = \theta,$$

$$\therefore \sin SK = \cos Y'' = \cos \phi \sin \theta.$$

Moreover, since AZ, AZ', are respectively perpendicular to the planes of xy and of $x'y'$, the inclinations of these two lines measure that of the planes; that is,

$$Z'' = \theta \therefore \cos Z'' = \cos \theta.$$

(273.) Having now determined the expressions for all the cosines which enter the formulas (A,) we have, by substituting them therein, the following new formulas for transforming the equation of any surface from one rectangular system of coordinates to another, viz.

$$x = x' (\cos \theta \sin \phi \sin \psi + \cos \phi \cos \psi)$$

$$+ y' (\cos \theta \sin \phi \cos \psi - \cos \phi \sin \psi)$$

$$- x' \sin \theta \sin \phi,$$

$$y = x' (\sin \phi \cos \psi - \cos \theta \cos \phi \sin \psi)$$

$$- y' (\cos \theta \cos \phi \cos \psi + \sin \phi \sin \psi)$$

$$+ x' \sin \theta \cos \phi,$$

$$z = x' \sin \theta \sin \psi + y' \sin \theta \cos \psi + z' \cos \theta.$$

(274.) These formulas become much more simple, when $\psi = 0$, that is, when the axis of x' coincides with the trace AR; for, since in this case, $\sin \psi = 0$, and $\cos \psi = 1$, the formulas become

$$x = x' \cos \phi + y' \sin \phi \cos \theta - x' \sin \theta \sin \phi$$

$$y = x' \sin \phi - y' \cos \phi \cos \theta + z \sin \theta \cos \phi$$

$$z = y' \sin \theta + x' \cos \phi.$$

CHAPTER III.

ON THE SECTIONS OF SURFACES OF THE SECOND ORDER.

Of Intersecting Planes.

(275.) The equation of a surface being given, let it be required to determine the equation of the intersection made by a plane whose inclination to, and trace on the plane of xy is given.

Call the inclination θ , and the angle formed by the trace and axis of x , ϕ ; then, calling the trace the axis of x' , and a perpendicular to it from the origin, and in the cutting plane, the axis of y' ; any point in the curve of intersection referred to these axes will be represented by $(x', y', 0)$ and the same point referred to the original axes of the surface is (x, y, z) ; hence, by putting $x' = 0$, in the formulas above, we have

$$x = x' \cos \phi + y' \sin \phi \cos \theta,$$

$$y = x' \sin \phi - y' \cos \phi \cos \theta,$$

$$z = y' \sin \theta;$$

and these expressions substituted in the equation of the surface will give the equation of the curve of intersection, when related to the axes of x' and y' , taken as directed in the cutting plane.

The values of the angles θ and ϕ are immediately determinable, when the equation of the plane is given. For, if this equation be

$$Ax + By + Cz + D = 0,$$

then (233)

$$\cos \theta = \frac{C}{\sqrt{A^2 + B^2 + C^2}},$$

and the equation of the trace on the plane of xy being (225)

$$Ax + By + D = 0,$$

we have

$$\tan \phi = -\frac{A}{B}.$$

Note. In the preceding transformations, we have supposed the origin to remain fixed; if, however, this be not the case, then the coordinates a, b, c , of the new origin must be introduced into the expressions.

(276.) Since the foregoing expressions for x, y, z are linear functions of x', y' , it follows that, when they are substituted in the equation of any surface of the second order, the result will be an equation also of the second order, between x' and y' ; hence every section of a surface of the second order is always a curve of the second order; and, indeed, whatever be the order of the surface, no higher can be the order of the curve of intersection; it follows, moreover, that a straight line cannot cut a surface of the n th order in more than n points.

Let us suppose a conical surface: If a plane cut one sheet through, the section will obviously be a curve returning into itself; and as we know it must be of the second order, we immediately conclude that the section must be an ellipse. If the cutting plane be parallel to the generating line, in any position, then the plane can obviously meet only one sheet of the surface; the section will therefore consist of but one branch; hence it can be no other curve than the parabola. If the plane be parallel to the axis of the cone, then both sheets will be cut, and the section will consist of two branches, and these will become two intersecting straight lines, when the axis coincides with the cutting plane; hence the section must be either an hyperbola, or one of its

varieties. On these accounts the lines of the second order are called *Conic Sections*. But the cone is not the only surface whose different sections furnish all the curves of the second order, as we shall presently see.

PROBLEM I.

(277.) To determine the nature of the different sections of a central surface of the second order.

In the general equation

$$Nx^2 + My^2 + Lz^2 = P \dots (1),$$

which comprehends the ellipsoid and hyperboloid, substitute for x, y, z , the values

$$x = x \cos \phi + y \cos \theta \sin \phi + a,$$

$$y = x \sin \phi - y \cos \theta \cos \phi + b,$$

$$z = y \sin \theta + c,$$

and the equation (1) then becomes restricted to the points in a plane section, and is reduced to

$$Ay^2 + Bry + Cx^2 + Dy + Ex + F = 0 \dots (2),$$

the coefficients of this equation having the following values:

$$A = L \sin^2 \theta + M \cos^2 \theta \cos^2 \phi + N \cos^2 \theta \sin^2 \phi,$$

$$B = 2(N - M) \cos \theta \sin \phi \cos \phi,$$

$$C = M \sin^2 \phi + N \cos^2 \phi,$$

$$D = 2(Lc \sin \theta - Mb \cos \theta \cos \phi + Na \cos \theta \sin \phi),$$

$$E = 2(Mb \sin \phi + Na \cos \phi),$$

$$F = Lc^2 + Mb^2 + Na^2 - P.$$

Now we know (158) that the curve represented by the equation (2) will be an ellipse, an hyperbola, or a parabola, according as $B^2 - 4AC$ is negative, positive, or 0. The value of one-fourth of this expression is

$$-MN \cos^2 \theta - LM \sin^2 \theta \sin^2 \phi - LN \sin^2 \theta \cos^2 \phi \dots (3).$$

Hence, if the surface is an ellipsoid, that is, if L, M, N , are all positive, $B^2 - 4AC$ must be negative; consequently, every section of an ellipsoid made by a plane is an ellipse, or else one of its varieties.

If the surface is an hyperboloid of one sheet, then of the coefficients L, M, N , two are positive, and the third negative; but, if the hyperboloid have two sheets, then two of the coefficients are negative, and one positive; so that the expression for $B^2 - 4AC$ must consist either of two positive terms and one negative, or else of two negative terms and one positive. In either case the aggregate of the terms may be either positive, negative, or 0. For, dividing each term of (3) by $\sin^2 \theta$, and abstracting from the signs, these terms may be represented by

$$Q \cot^2 \theta, \quad R \sin^2 \phi, \quad S \cos^2 \phi,$$

and it is plain that an infinite variety of values may be given to ϕ and θ , that will render possible either of the conditions

$$\left. \begin{aligned} Q \cot^2 \theta &> (R \sin^2 \phi + S \cos^2 \phi), \\ Q \cot^2 \theta &< (R \sin^2 \phi + S \cos^2 \phi) \end{aligned} \right\} \dots (4),$$

$$\left. \begin{aligned} Q \cot^2 \theta &= (R \sin^2 \phi + S \cos^2 \phi), \\ Q \cot^2 \theta &= (R \sin^2 \phi - S \cos^2 \phi) \end{aligned} \right\} \dots (5),$$

for $\cot \theta$ may be made any value we please, from 0 to ∞ .

By the conditions (4) it appears that any term may be made to exceed numerically the sum of the other two, and, consequently, the aggregate of the three terms may take the sign of any one of

them; that is, it may be either positive or negative. The conditions (5) show that either term may become equal to the sum of the other two; so that, whichever two have the same sign, their aggregate may become equal to the third, when the aggregate of the whole will be 0.

Hence *the section of an hyperboloid by a plane may, like the cone which is a variety of it, be either an ellipse, an hyperbola, or a parabola.*

Cor. As none of the constants a, b, c , enter into the expression (3), the curve of intersection must continue of the same kind, however these constants may be altered, provided only that the angles θ, ϕ , remain the same; that is, *parallel sections always give the same kind of curve.*

PROBLEM II.

(278.) To determine the nature of the sections in surfaces which have not a centre.

Substituting the formulas employed in last problem, in the general equation

$$My^2 + Lx^2 = Qx \dots (1),$$

we obtain a result of the form

$$Ay^2 + Bxy + Cx^2 + \&c. = 0,$$

in which

$$A = L \sin^2 \theta + M \cos \theta \cos^2 \phi,$$

$$B = -2M \cos \theta \sin \phi \cos \phi,$$

$$C = M \sin^2 \phi,$$

$$\therefore B^2 - 4AC = 4LM \sin^2 \theta \sin^2 \phi \dots (2).$$

If the surface is the elliptic paraboloid, then the coefficients L, M , being of the same sign, the expression for $B^2 - 4AC$ must

be negative, unless $\theta = 0$, or $\phi = 0$, when the expression becomes $= 0$. Hence the intersections of an elliptic paraboloid must be either ellipses or parabolas, or else varieties of these curves.

If the paraboloid is hyperbolic, then L, M , having contrary signs, the expression (2) can never be positive; so that the intersections of the parabolic hyperboloid must be either hyperbolas or parabolas, or else varieties of these curves.

Cor. It follows here, as in last problem, that because the expression (2) is independent of the values of the constants a, b, c , parallel sections always give the same kind of curve.

PROBLEM III.

(279.) To determine the locus of the centres of any parallel sections.

Let the surface be central; then, when the sections are central curves, let us suppose the coordinates in each to originate at the centre; and let us represent by (x', y', z') the centre of any parallel section when referred to the axes to which the surface is referred; then x', y', z' mean the same thing here as a, b, c , in the preceding problems. Now, since the axes of the section are here supposed to originate at the centre, the terms Dy, Ex , in equation (2), prob. 1. will be absent (160), showing that, in this case, we must have

$$D = 2(Lx' \sin \theta - My' \cos \theta \cos \phi + Nz' \cos \theta \sin \phi) = 0,$$

$$E = 2(My' \sin \phi + Nz' \cos \phi) = 0.$$

These equations being linear, each separately represents a plane; but, as they exist together, they denote the straight line, which is their intersection; hence the locus of (x', y', z') is a straight line.

If the surface have no centre, we should, in the same manner, find that, when the parallel sections have centres, these are all situated in the same straight line.

Neither of the preceding equations having a constant term, it follows (225) that the planes which they represent both pass through the origin; this point then belongs to their intersection. Hence *the centres of parallel sections are all on the same diameter of the surface.*

On the Circular Sections of Surfaces of the Second Order.

(280.) It is obvious that all surfaces of revolution, of whatever order, may be cut by a plane so as to give a circular section; since every point in the generating line describes a circle whose centre is upon, and whose plane is perpendicular to, the axis of revolution. It is interesting to enquire whether it is *essential* that a surface be one of revolution, in order that a circular section of it may be possible; in other words, whether the possibility of a circular section marks the surface as one of revolution. We shall examine this point as regards the surfaces of the second order.

(281.) *The central surfaces of the second order are all comprised in the equation*

$$Nx^2 + My^2 + Lz^2 = P \dots (1),$$

and the section of one of these by a plane, inclined to that of xy in the angle of θ , and of which the trace makes the angle ϕ with the axis of x , is generally denoted by the equation

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0,$$

and our business now is to ascertain whether it is possible to give the cutting plane such a direction, that is, to determine such values for the arbitrary quantities θ and ϕ , that this equation must represent a circle, even though the coefficients of the variables in the equation (1) be all unequal. In other words, can any values for these quantities be found that will render the conditions

$$B = 0, A = C$$

possible? The values of A, B, C, in terms of the quantities referred to, have been already given in (277); the equations of condition are therefore

$$(N - M) \cos \theta \sin \phi \cos \phi = 0$$

$$L \sin^2 \theta + M \cos^2 \theta \cos^2 \phi + N \cos^2 \theta \sin^2 \phi =$$

$$M \sin^2 \phi + N \cos^2 \phi.$$

The first of these is satisfied by either of the three values

$$\cos \theta = 0, \sin \phi = 0, \cos \phi = 0;$$

which values substituted successively in the second equation give, for the remaining quantity, the several correlative expressions which follow, viz.

$$\cos \theta = 0 \text{ accompanies } \tan \phi = \pm \sqrt{\frac{L - N}{M - L}}$$

$$\sin \phi = 0 \quad \tan \theta = \pm \sqrt{\frac{N - M}{L - N}}$$

$$\cos \phi = 0 \quad \tan \theta = \pm \sqrt{\frac{M - N}{L - M}}.$$

Now since the values of L, M, N are unequal,—as they necessarily must be when the surface is not of revolution,—which we here assume, because when it is of revolution there is no question about the possibility of circular sections; we say since these values are all different, *one*, and only one, of the preceding expressions can be *real*:—the other two must be *imaginary*. That *one* must be real is immediately seen by multiplying all three together; for the product, independently of the radical, is

$$\frac{(M - N)^2}{(L - M)^2}$$

a quantity that is essentially positive, which however could not be unless one, at least, of the three factors is itself positive.

That more than one of these factors cannot be positive, will be seen from observing that whichever we suppose to have *like* signs in numerator and denominator, the two others will each have the numerator and denominator with *unlike* signs. Suppose for instance the first expression to be real or to have the numerator and denominator under the radical alike as to sign. This condition requires that we must have

$$N > L > M$$

or else

$$M > L > N.$$

The first hypothesis causes the numerator of the second fraction to be *plus*, and its denominator to be *minus*; and it causes the numerator of the third fraction to be *minus*, and its denominator to be *plus*. The signs are obviously reversed in the second hypothesis; and in like manner if either of the other expressions be supposed possible, the remaining two will necessarily be impossible.

It follows therefore, since for each of the values $\cos \theta = 0$, or $\sin \phi = 0$, or $\cos \phi = 0$, there corresponds *two* values for the tangent of the other angle, that *through each point of a central surface of the second order two different planes may be drawn so as to give a circular section*.

These two sections merge into one when the surface is of revolution. For if any two of the coefficients L, M, N be equal, *one* only of the foregoing expressions will be imaginary; and the other two will become identical, furnishing that single position for the intersecting plane indicated by $\cos \theta = 0$, $\tan \phi = \infty$, or, which is the same thing, $\cos \phi = 0$.

We may remark that the condition $\cos \theta = 0$ indicates that the secant plane is perpendicular to that of xy ; so that when the coefficients of the variables in the equation (1), of the surface, have such values as to render the expression above for $\tan \phi$, possible, the two circular sections, through any point, will be each perpendicular to that principal section of the surface which is taken for

the plane of xy ; and their traces will be equally inclined to that principal diameter which is taken for the axis of x . In the central surfaces of revolution this inclination is, as we have just seen, 90° for both planes; so that the two traces are confounded in a single trace, parallel to the axis of y .

When the coefficients of the proposed equations are such as to render possible one of the other expressions above, then the planes of the circular sections must be perpendicular to the plane of yz , or to that of xz , according as the second or third of the expressions referred to are possible.

(282.) *The non-central surfaces of the second order are all represented by the form*

$$My^2 + Lz^2 = Qx,$$

and, in order that the section through any point in such surface may be circular, we must, as before, have the following conditions among the leading coefficients in the equation of that section; viz. $B = 0$, $A = C$; that is (278) we must have

$$2M \cos \theta \sin \phi \cos \phi = 0$$

$$L \sin^2 \theta + M \cos^2 \theta \cos^2 \phi = M \sin^2 \phi.$$

The supposition $\sin \phi = 0$ is here obviously forbidden; for the section due to this inclination, and fulfilling the required conditions, will be represented by an equation in which A , B , and C are absent; and cannot therefore be a *curve* at all. Seeing therefore that, for a circular section to exist, $\sin \phi$ must have some value, the first condition merely requires that the equation

$$\cos \theta \cos \phi = 0$$

be satisfied; which reduces the second equation to

$$L \sin^2 \theta = M \sin^2 \phi,$$

and these are satisfied for the two systems of simultaneous values following, and for these only: viz.

$$\cos \theta = 0, \sin \phi = \pm \sqrt{\frac{L}{M}},$$

and

$$\cos \phi = 0, \sin \theta = \pm \sqrt{\frac{M}{L}};$$

systems which are both impossible when L and M have opposite signs, as in the case of the *hyperbolic paraboloid*; a surface which we already know to be incapable of furnishing any but hyperbolic or parabolic sections (278).

When L and M have *like* signs then *one* of these systems is admissible, but only one;—not that in such a case either expression becomes *imaginary*, but because one of these expressions must be greater than unity; and therefore cannot be the value of any sine. Hence *through any point in a non-central surface of the second order, the hyperbolic paraboloid excepted, two planes furnishing circular sections may be drawn.*

When the surface is of revolution, $L = M$; and both systems give one and the same plane, fixed by the values $\sin \theta = \pm 1$, $\sin \phi = \pm 1$, showing it to be perpendicular to the axis of x . But when the surface is not of revolution, and $L < M$, the two planes have their traces equally inclined to this axis, and are themselves perpendicular to that principal section of the surface in which the axes of x and of y are situated. If $M < L$ the sections have their traces perpendicular to the axis of x , and are themselves equally inclined to the plane of xy .

(283.) As in both kinds of surfaces the inclinations θ and ϕ alone determine the position of the plane competent to give a circular section, and as these inclinations are the same in parallel planes, it follows that all the sections, parallel to any circular section, are themselves circular. In the ellipsoid these sections diminish as the plane recedes from the centre; and the circle merges into a point when the plane becomes a tangent. There are evidently

four of these points on the surface of the ellipsoid,—points at which the tangent planes are parallel to all the circular sections; they are symmetrically situated, being at the extremities of two diameters (279) equally inclined to the principal diameters. They differ from all other points on the surface; being those at which the surface has a greater uniformity of curvature than anywhere else; as, from the nature of the sections which these points terminate, it is clear that, in their immediate vicinity, the surface is more nearly spherical than in any other part, seeing that elsewhere the sections are always elliptical. These remarkable points are those called by the French mathematicians "*ombilics*," or *umbilical points*. The hyperboloid of two sheets, like the ellipsoid, has *four* of these points; but, when either surface is of revolution, a pair merges into a single one, in consequence of the two diameters, at the extremities of which they are placed, becoming confounded by both coinciding with the axis of revolution; so that, if the surface be one of revolution, the only umbilical points are those at the extremities of that diameter which is the axis of revolution.

A little reflection on the part of the student will enable him to perceive that the two symmetrically disposed sections which pass through the centre of the ellipsoid have the *mean axis* of the surface for their common intersection. The tangent planes parallel to these must therefore have their points of contact on the section perpendicular to this axis; that is, on the section through the major and minor principal diameters: on this section, therefore, the *umbilical points* are always situated. If the minor diameter be supposed gradually to lengthen while the other two remain constant, the central sections will gradually approach each other, each becoming more nearly perpendicular to the section containing the mean and major diameters, the umbilical points at the same time approaching the vertices of the latter; till at length,—when the minor diameter becomes equal to the mean, and the surface passes into one of revolution,—the two central sections both merge into that perpendicular to the axis of revolution; and the two umbilical points that have hitherto been approaching one extremity of the major diameter actually meet, and become confounded in that extremity; while

the remaining two unite in like manner at the opposite extremity. If, on the other hand, the major diameter should shorten till it become equal to the mean, the central sections would separate more and more from the perpendicular to the shortening axis, and would both pass through it and coincide when the major axis became equal to the mean. The umbilical points, which had all along been advancing towards the opposite ends of the minor axis, would then merge into them; and thus, as in the former case, become confounded with the poles of the surface.

As to the other surfaces, it is plain that the elliptic paraboloid has but *two* umbilical points, which coalesce when the surface is one of revolution; and that the hyperbolic paraboloid has no such point, for it has no circular sections. The hyperboloid of one sheet is also without umbilical points; for, although it has two series of circular sections, yet each series terminates not in a point, but in a *pair of intersecting straight lines*; it being a remarkable property of this surface, as also of the hyperbolic paraboloid, that every tangent plane touches the surface along two straight lines. For the proof of this and of various other interesting properties, see the *Differential Calculus*, Section III.

CHAPTER III.

DISCUSSION OF THE GENERAL EQUATION.

(284.) We shall now proceed to examine the equation of the second degree of three variables, in its most general form; and show that it can never represent any surface not among those which we have already examined. This general equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Kz + L = 0 \quad (1),$$

and we shall for simplicity suppose it to refer the surface which it represents to rectangular axes. This supposition will not in the least diminish the generality of our reasoning; since, if the axes were originally oblique, they might be transformed to rectangular, by the substitution of certain *linear* functions of the new coordinates in place of the old, so that the *degree* of the equation would remain the same and its *generality* could not exceed that of equation (1).

Let us now transform these rectangular axes to another system, also rectangular, by substituting, in equation (1), the values (269)

$$\left. \begin{aligned} x &= x' \cos X + y' \cos X' + z' \cos X'' \\ y &= x' \cos Y + y' \cos Y' + z' \cos Y'' \\ z &= x' \cos Z + y' \cos Z' + z' \cos Z'' \end{aligned} \right\} \dots (A),$$

then the resulting equation must be of the form

$$A'x'^2 + B'y'^2 + C'z'^2 + D'x'y' + E'x'z' + F'y'z' + G'x' + H'y' + K'z' + L = 0 \dots (2),$$

in which the coefficients are functions of the nine angles which enter the formulas (A).

Now it has been seen (269) that these nine angles are subject to only six conditions; and that, therefore, in order to fix their values, three more conditions must be introduced among them; and the only limit to the choice of these conditions is that they must not be inconsistent with the other six.

Let us here suppose the three conditions

$$D' = 0, E' = 0, F' = 0 \dots (B),$$

then, if it can be shown that these may exist conjointly with the conditions (1) and (3), art. (270), we may immediately infer that the general equation (1) may always be reduced to the more simple form

$$A'x'^2 + B'y'^2 + C'z'^2 + G'x' + H'y' + K'z + L = 0,$$

by merely altering the directions of the rectangular axes.

The expressions for the coefficients D' , E' , F' , may be obtained without substituting the expressions (A) in every term of the equation (1):—the last four terms obviously have no influence on these coefficients; and, instead of actually squaring the expressions (A) for the first three terms, we need only attend to the products two and two of the three terms, in each, these products being the only parts of the squares concerned in the formation of the three coefficients under consideration. Also in the three following terms of equation (1), which contain the products of the expressions (A), two and two, the partial products, arising from multiplying any term by that in the same vertical row, are not concerned in these coefficients; and are, therefore, not to be attended to. Availing ourselves of these considerations, we find for the terms $F'x'y'$, $E'x'z'$, $D'x'y'$, the expressions

$$\begin{array}{l} 2A \cos Z \cos Z' \quad x'y' + 2A \cos Z \cos Z'' \quad x'z' + 2A \cos Z' \cos Y'' \quad x'y' \\ + 2B \cos Y \cos Y' \quad + 2B \cos Y \cos Y'' \quad + 2B \cos Y' \cos Y'' \\ + 2C \cos X \cos X' \quad + 2C \cos X \cos X'' \quad + 2C \cos X' \cos X'' \\ + D \cos Z \cos Y' \quad + D \cos Z \cos Y'' \quad + D \cos Z' \cos Y'' \\ + D \cos Y \cos Z' \quad + D \cos Y \cos Z'' \quad + D \cos Y' \cos Z'' \\ + E \cos Z \cos X' \quad + E \cos Z \cos X'' \quad + E \cos Z' \cos X'' \\ + E \cos X \cos Z' \quad + E \cos X \cos Z'' \quad + E \cos X' \cos Z'' \\ + F \cos Y \cos X' \quad + F \cos Y \cos X'' \quad + F \cos Y' \cos X'' \\ + F \cos X \cos Y' \quad + F \cos X \cos Y'' \quad + F \cos X' \cos Y'' \end{array}$$

Hence the three equations of condition are

$$\begin{array}{l} 2A \cos Z \quad \cos Z' + 2B \cos Y \quad \cos Y' + 2C \cos X \quad \cos X' = 0 \\ + D \cos Y \quad + D \cos Z \quad + E \cos Z \\ + E \cos X \quad + F \cos X \quad + F \cos Y \end{array}$$

$$\begin{array}{l} 2A \cos Z \left| \begin{array}{l} \cos Z'' + 2B \cos Y \\ + D \cos Y \\ + E \cos X \end{array} \right| \begin{array}{l} \cos Y'' + 2C \cos X \\ + E \cos Z \\ + F \cos Y \end{array} \left| \cos X'' = 0 \right. \end{array}$$

and

$$\begin{array}{l} 2A \cos Z' \left| \begin{array}{l} \cos Z'' + 2B \cos Y' \\ + D \cos Y' \\ + E \cos X' \end{array} \right| \begin{array}{l} \cos Y'' + 2C \cos X' \\ + E \cos X' \\ + F \cos Y' \end{array} \left| \cos X'' = 0 \right. \end{array}$$

or more briefly

$$\left. \begin{array}{l} M \cos Z' + N \cos Y' + P \cos X' = 0 \\ M \cos Z'' + N \cos Y'' + P \cos X'' = 0 \\ M' \cos Z'' + N' \cos Y'' + P' \cos X'' = 0 \end{array} \right\} \dots \dots (B).$$

These, then, are the equations which must exist in conjunction with the following, if the transformation in view is possible,

$$\left. \begin{array}{l} \cos X \cos X' + \cos Y \cos Y' + \cos Z \cos Z' = 0 \\ \cos X \cos X'' + \cos Y \cos Y'' + \cos Z \cos Z'' = 0 \\ \cos X' \cos X'' + \cos Y' \cos Y'' + \cos Z' \cos Z'' = 0 \end{array} \right\} \dots \dots (C)$$

$$\left. \begin{array}{l} \cos^2 X + \cos^2 Y + \cos^2 Z = 1 \\ \cos^2 X' + \cos^2 Y' + \cos^2 Z' = 1 \\ \cos^2 X'' + \cos^2 Y'' + \cos^2 Z'' = 1 \end{array} \right\} \dots \dots (D)$$

Eliminating N from the first two of equations (B), by multiplying the first by $\cos Y''$, and the second by $\cos Y'$, and then, subtracting the first result from the second, we get

$$\left. \begin{array}{l} M (\cos Y' \cos Z'' - \cos Z' \cos Y'') \\ + P (\cos Y' \cos X'' - \cos X' \cos Y'') \end{array} \right\} = 0 \dots \dots (E);$$

eliminating P from the same equations, we have

$$\left. \begin{aligned} M (\cos X' \cos Z'' - \cos Z' \cos X'') \\ + N (\cos X' \cos Y'' - \cos Y' \cos X'') \end{aligned} \right\} = 0 \dots (F).$$

In like manner, by eliminating first $\cos Y$, and then $\cos X$, from the first two of equations (C) we have

$$\left. \begin{aligned} \cos X (\cos Y' \cos X'' - \cos X' \cos Y'') \\ + \cos Z (\cos Y' \cos Z'' - \cos Z' \cos Y'') \end{aligned} \right\} = 0 \dots (G),$$

and

$$\left. \begin{aligned} \cos Y (\cos X' \cos Y'' - \cos Y' \cos X'') \\ + \cos Z (\cos X' \cos Z'' - \cos Z' \cos X'') \end{aligned} \right\} = 0 \dots (H).$$

Putting, for simplicity,

$$\cos Y' \cos Z'' - \cos Z' \cos Y'' = Q,$$

$$\cos Y' \cos X'' - \cos X' \cos Y'' = R,$$

$$\cos X' \cos Z'' - \cos Z' \cos X'' = S,$$

the four preceding equations become

$$MQ + PR = 0, \quad MS - NR = 0,$$

$$Q \cos Z + R \cos X = 0, \quad S \cos Z - R \cos Y = 0,$$

in which the quantities Q, R, S, are the only ones containing the accented cosines. These three quantities may be eliminated from the four equations thus: the first and second give

$$Q = -\frac{PR}{M}, \quad S = \frac{NR}{M},$$

which values, substituted in the remaining two, give

$$P \cos Z - M \cos X = 0, \quad N \cos Z - M \cos Y = 0,$$

or, replacing P , M , and N , by their values, these equations are

$$\left. \begin{aligned} (2C \cos X + E \cos Z + F \cos Y) \cos Z \\ - (2A \cos Z + D \cos Y + E \cos X) \cos X \end{aligned} \right\} = 0 \dots (I.)$$

$$\left. \begin{aligned} (2B \cos Y + D \cos Z + F \cos X) \cos Z \\ - (2A \cos Z + D \cos Y + E \cos X) \cos Y \end{aligned} \right\} = 0 \dots (K.)$$

Now these two equations, together with

$$\cos^2 X + \cos^2 Y + \cos^2 Z = 1$$

are sufficient to determine the three angles, $\cos X$, $\cos Y$, $\cos Z$. For, put

$$m = \frac{\cos X}{\cos Z}, \text{ and } n = \frac{\cos Y}{\cos Z} \therefore \cos Z = \frac{1}{\sqrt{1 + m^2 + n^2}}$$

then, dividing the first and second equations by $\cos^2 Z$, they become

$$2(C - A)m + E(1 - m^2) - Dmn + Fn = 0,$$

$$2(B - A)n + D(1 - n^2) - Emn + Fm = 0.$$

From this last we get

$$\frac{2(B - A)n + D(1 - n^2)}{En - F} = m,$$

which value of m , substituted in the preceding equation, gives, after reduction, a cubic equation; for, although the second term of the equation will furnish a term containing the fourth power of n , viz. $D^2 En^4$, yet, when the whole result is multiplied by $(En - F)^2$, to clear it of fractions, this term will obviously be also given, with contrary sign, in the value of Dmn , and is thus destroyed. This being a cubic equation, there necessarily exists at least one real value for n , and, consequently, the value of m is real; and hence

also the values of $\cos Z$, of $\cos X = m \cos Z$, and of $\cos Y = n \cos Z$. We have thus proved the reality of the three cosines, $\cos X$, $\cos Y$, $\cos Z$.

If we now go back to the equations (B) and (C), and proceed with the first and third of each group, exactly as we have done with the first and second, taking care, however, to put the first of (B) under the form

$$M' \cos Z + N' \cos Y + P' \cos X = 0,$$

to which it is obviously identical; we shall, in the same way, establish the reality of $\cos X'$, $\cos Y'$, $\cos Z'$; and lastly, employing the second and third equations of each group, we may demonstrate the reality of $\cos X''$, $\cos Y''$, $\cos Z''$.

Hence we may infer that it is always possible to reduce the general equation (1) to the form

$$A'x^2 + B'y^2 + C'z^2 + G'x + H'y + K'z + L = 0 \dots\dots (1'),$$

by altering the position of the rectangular axes to which the surface represented by it is referred.

(285.) We shall now show that the equation in this form may be finally reduced to one or other of the more simple forms

$$A'x^2 + B'y^2 + C'z^2 + P = 0 \dots\dots (A').$$

$$B'y^2 + C'z^2 + G'x = 0$$

$$C'x^2 + G'x + K'y = 0.$$

For, let the origin be removed, by substituting in (1') the values

$$x = x' + a, \quad y = y' + b, \quad z = z' + c,$$

then, putting P for the last term, it becomes

$$A'x'^2 + B'y'^2 + C'z'^2 + 2A'a \left| \begin{array}{c} x' \\ + G' \end{array} \right| + 2B'b \left| \begin{array}{c} y' \\ + H' \end{array} \right| + 2C'a \left| \begin{array}{c} z' \\ + K' \end{array} \right| + P = 0,$$

in which the coefficients of x' , of y' , and of z' vanish when we have the conditions

$$c = -\frac{G'}{2A'}, \quad b = -\frac{H'}{2B'}, \quad a = -\frac{K'}{2C'},$$

which are always possible when the coefficients A' , B' , C' , are neither of them 0. But, if one of these coefficients be 0, then it will not be possible to remove the variable, whose square is absent; thus, if $A' = 0$, then, that the coefficient of x' may be 0, there must be $c = -\frac{G'}{0}$, that is, the new origin must be infinitely distant from the old,

in the direction of the axis of x ; this origin, therefore, is not determinable. Nothing, however, hinders us from removing the two terms in y' and z' , and thus reducing the equation to the form

$$B'y^2 + C'x^2 + G'z + P = 0 \dots (B'),$$

and, as the quantity c enters P , and is still arbitrary, we may determine it from the condition $P = 0$, which will finally reduce the equation to the second of the above form, viz. to

$$B'y^2 + C'x^2 + G'z = 0 \dots (C').$$

But, if not only $A' = 0$, but also $G' = 0$, in equation (1'), that is, if one of the variables z be entirely absent from the equation, then, (B') is simply

$$B'y^2 + C'x^2 + P = 0,$$

in which case the surface is obviously (240) a cylinder, whose base or directrix on the plane of xy is either an ellipse or hyperbola, according as the signs of B' and C' are like or unlike.

If two of the coefficients A' , B' , C' , are 0, as $A' = 0$, $B' = 0$, then the removal of the terms, containing the first powers of the variables whose squares are absent, is impossible; but the conditions $a = -\frac{K'}{2C'}$, and $P = 0$, may still exist, and will reduce the equation to

$$C'x^2 + G'z + K'y = 0.$$

When all three of the squares are absent from (1), the equation represents a plane.

(286.) From the preceding discussion it follows that any surface of the second order may be represented by one or other of the following equations, viz.

$$Lx^2 + My^2 + Nz^2 + P = 0$$

$$Lx^2 + My^2 + Qz = 0$$

$$Lx^2 + My^2 + P = 0$$

$$Lx^2 + Gz + Hy = 0.$$

All the surfaces represented by the first two of these equations have been fully considered in Chapter III. They were found to comprehend ellipsoids; hyperboloids of one and of two sheets; paraboloids, elliptic and hyperbolic; and, as varieties of these, the sphere and the cone.

With regard to the remaining two equations, the first we have seen characterizes cylindrical surfaces, whose directrices on the plane of xy are either ellipses or hyperbolas. The other equation is also that of a cylindrical surface; but of this the directrix is a parabola: for, suppose it to be cut by a series of planes parallel to the plane of xy , that is, by planes of which the equations are

$$x = k, \quad x = k', \quad x = k'', \text{ \&c.}$$

then for the sections we have the equations

$$Gz + Hy = -Lk^2,$$

$$Gz + Hy = -Lk'^2,$$

$$Gz + Hy = -Lk''^2, \text{ \&c.};$$

and these all representing parallel straight lines, it follows that we may conceive the surface to be generated by a straight line moving parallel to itself. This surface must, therefore, be a cylinder. For the directrix or trace on the plane of xy put $z = 0$, in its equation, and we have

$$Lx^2 + Hy = 0,$$

which represents a parabola, or one of its varieties.

(287.) We may obviously infer from this discussion that the only conical surface of the second order is the elliptic cone, of which the circular is a variety; in other words, this conical surface is always such that a system of coordinate planes may be found that will render the trace on the plane of xy an ellipse or a circle. But we may assume any curve of the second order on the plane of xy , and thus, agreeably to art. (242), generate a conical surface, which will also be of the second order. It follows, therefore, that by giving different inclinations to the plane of xy , the elliptic cone will furnish for traces on that plane all the curves of the second order, as was also shown from other considerations at art. (276).

(288.) We shall now briefly discuss the general equation (1), in order to ascertain criteria by which we may know, without the trouble of transforming it, the nature of the surface which it represents.

Solving the equation for z , we have, by putting Q for the quantity under the radical,

$$z = -\frac{Dy + Ex + G}{2A} \pm \frac{1}{2A} \sqrt{Q}.$$

Also, solving for y , we have

$$y = -\frac{Ds + Fx + H}{2B} \pm \frac{1}{2B} \sqrt{Q};$$

and, in like manner, for x ,

$$x = -\frac{Es + Fy + K}{2C} \pm \frac{1}{2C} \sqrt{Q''}.$$

Representing the rational parts of these expressions by Z , Y , and X , respectively, we have the three equations,

$$Z = -\frac{Dy + Ex + G}{2A},$$

$$Y = -\frac{Ds + Fx + H}{2B},$$

$$X = -\frac{Es + Fy + K}{2C},$$

representing three planes which, from the foregoing general expressions for s , y , and x , are obviously such that they bisect the chords drawn parallel to the axes of z , y , x ; each of these planes therefore, passes through the centre; that is, it is diametral; so that the centre of the surface, supposing it to have one, must be at the intersection of these planes. Calling this intersection (x' , y' , z') the preceding equations give

$$\left. \begin{aligned} 2Ax' + Dy' + Ez' + G &= 0, \\ 2By' + Dx' + Fz' + H &= 0, \\ 2Cx' + Es' + Fy' + K &= 0, \end{aligned} \right\} \dots (1),$$

from which we get, by elimination, the three expressions for x' , y' , and z' . The common denominator of these expressions we find to be

$$AF^2 + BE^2 + CD^2 - DEF - 4ABC;$$

hence, if this be finite, the surface will have a centre; but if it be $= 0$, then the surface will not have a centre; that is:—

If the surface have a centre,

$$AF^2 + BE^2 + CD^2 - DEF - 4ABC \geq 0 \dots (2).$$

If the surface have not a centre,

$$AF^2 + BE^2 + CD^2 - DEF - 4ABC = 0 \dots (3).$$

If $G = 0$, $H = 0$, and $K = 0$, then (*Algebra*, p. 78) the numerators of the expressions for x' , y' , z' , each become 0; hence, if in this case the condition (3) do not exist, the coordinates of the centre each become 0; that is, the centre of the surface must coincide with the origin, as is also at once manifest from the equations (1) themselves, which represent planes passing through the origin when the constants G , H , K , are absent; therefore, whatever be the inclinations of the axes, provided they originate at the centre of the surface, the most general form of the equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + L = 0.$$

If, however, the condition (3) exists at the same time that we have $G = 0$, $H = 0$, $K = 0$, then the expressions for (x', y', z')

become each $= \frac{0}{0}$, intimating that there are an indefinite number

of centres, or points, in which the three planes (1) meet; hence they must intersect in one common straight line, passing through the origin, and which is the locus of all the centres of the surface. This surface having innumerable centres in the same straight line can be no other than an elliptic or hyperbolic cylinder.

(289.) Having given criteria (2) and (3) for discovering when the equation represents a central surface, and when it does not, let us now enquire by what means we may know after a surface is ascertained to be central, whether it is limited or unlimited, that is, whether it belongs to the class of ellipsoids or of hyperboloids.

If the surface be limited in every direction, then to whatever point in the surface a line from the origin be drawn, this line will always have a finite length; but, should the surface be unlimited in any direction, then, as some of its points will be infinitely distant from the origin, lines drawn to them from the origin will not be finite; and it is hence obvious that while, in the former case, every section of the surface is a limited curve, every section, in the latter case, which passes through either of the infinite lines which we have supposed to be drawn, will necessarily be an unlimited curve. This being the case with every such section, we need consider only those made by planes perpendicular to one of the coordinate planes, when, if it is found that no one of these can possibly give an unlimited curve, we may conclude that the surface is itself limited in every direction.

Now the equation of any plane drawn perpendicularly to the plane of xz , and passing through the origin, is

$$x = ax.$$

Combining this with the equation of the surface (284), we have for the intersection the equation

$$(A + Ca^2 + Ea)x^2 + (D + Fa)xy + By^2 + (G + Ka)x + Hy + L = 0,$$

and, in order that this may represent always a limited curve, we must have (158)

$$(D + Fa)^2 - 4(A + Ca^2 + Ea)B < 0 \dots (4).$$

whatever be the value of a ; that is,

$$(F^2 - 4BC)a^2 + 2(DF - 2BE)a + D^2 - 4AB < 0,$$

or, dividing by $F^2 - 4BC$, and then, decomposing the expression

into factors, $(\alpha - \beta)$, $(\alpha = \beta')$, (see art. 169), we have

$$(F^2 - 4BC)(\alpha - \beta)(\alpha - \beta') < 0.$$

Now this expression cannot preserve the same sign for every value of α , unless $(\alpha - \beta)(\alpha - \beta')$ does, and that the sign of this may always remain the same, that sign must be positive, and the values of β and β' imaginary (*Theory of Equations*, art. 15); hence, that the condition (4) may exist, we must have, in the first place, $F^2 - 4BC < 0$, and, in the second place, the roots β , β' , of the equation

$$\alpha^2 + \frac{2(DF - 2BE)}{F^2 - 4BC}\alpha + \frac{D^2 - 4AB}{F^2 - 4BC} = 0$$

must be imaginary; that is to say, the quantity under the radical which enters into the expressions for these roots, which quantity we find to be

$$(DF - 2BE)^2 - (F^2 - 4BC)(D^2 - 4AB),$$

must be negative.

Hence we may conclude that, when the general equation represents a surface limited in every direction, there must exist the relations

$$F^2 - 4BC < 0$$

$$(DF - 2BE)^2 - (F^2 - 4BC)(D^2 - 4AB) < 0.$$

CHAPTER IV.

MISCELLANEOUS PROPOSITIONS IN GEOMETRY OF
THREE DIMENSIONS.

PROPOSITION I.

(290.) To determine the distance between two points in space, when the axes of reference are oblique.

First, let one of the points be at the origin, and let the *rectangular* coordinates of the other point be x, y, z ; its *oblique* coordinates x', y', z' . Then (220)

$$D^2 = x^2 + y^2 + z^2.$$

For x, y, z in this expression substitute their values in terms of x', y', z' as given in equation (A) page 198; and the result, when account is taken of the conditions (1), (2), which must subsist in conjunction with (A), is easily seen to be

$$D^2 = x'^2 + y'^2 + z'^2 + 2x'y' \cos [x', y'] + 2x'z' \cos [x', z'] + 2y'z' \cos [y', z'].$$

This, then, is the expression for the square of the diagonal of an oblique parallelepiped whose contiguous edges are x', y', z' . Hence when these edges are represented by $x' - x'', y' - y'', z' - z''$, the square of the diagonal must be

$$\begin{aligned} D^2 = & (x' - x'')^2 + (y' - y'')^2 + (z' - z'')^2 \\ & + 2(x' - x'')(y' - y'') \cos [x', y'] \\ & + 2(x' - x'')(z' - z'') \cos [x', z'] \\ & + 2(y' - y'')(z' - z'') \cos [y', z']. \end{aligned}$$

This is, therefore, the square of the distance between any two points (x', y', z') and (x'', y'', z'') in space.

By means of this expression for D^2 , that for $\cos V$, the inclinations of two straight lines in space, when referred to oblique axes, may be found from the first of those for $\cos V$, in Prob. VIII. p. 129 ; but as this expression would be exceedingly complex, and of little use, we shall not exhibit it. The complication and inutility of the expressions for the inclination of two planes, or of a line and a plane, will also justify their omission, although we had intended to give them in this chapter.

We may here notice that the equation just deduced is that of a sphere of radius, D , when represented analytically in the most general manner, that is, without any restriction as to axes of reference.

PROPOSITION II.

(291.) To determine the length of the shortest distance between two straight lines in space.

If a plane be drawn through one of the lines and parallel to the other, it is evident that a perpendicular, from any point in this other to the plane, will measure the shortest distance between the two lines. Let then the equations of the two lines be

$$\left. \begin{array}{l} x = mz + a \\ y = nz + b \end{array} \right\} \text{ and } \left\{ \begin{array}{l} x = m'z + a' \\ y = n'z + b' \end{array} \right.$$

and let the equation of the plane, through the first and parallel to the second, be

$$z = Ax + By + C.$$

Then the coefficients in these equations must fulfil the conditions (227) of coincidence, and of parallelism, that is, we must have

$$Aa + Bb + C = 0, \quad Am + Bn = 1, \quad Am' + Bn' = 1,$$

from which we get

$$A = \frac{n' - n}{mn' - mn}, \quad B = \frac{m - m'}{mn' - m'n}, \quad C = -Aa - Bb.$$

Now the expression for the perpendicular from the point $(x, y, 0)$ in the second line to the plane is (231)

$$\begin{aligned} P &= \frac{-Aa' - Bb' - C}{\sqrt{1 + A^2 + B^2}} \\ &= \frac{-(n' - n)a' + (m' - m)b' + (n' - n)a - (m' - m)b}{\sqrt{(mn' - m'n)^2 + (n' - n)^2 + (m' - m)^2}} \\ &= \frac{(n' - n)(a - a') - (m' - m)(b - b')}{\sqrt{(mn' - m'n)^2 + (n' - n)^2 + (m' - m)^2}} \end{aligned}$$

which is the expression for the shortest distance required.

PROPOSITION III.

(292.) To prove that in a central surface of the second order there is an infinite number of systems of conjugate diameters.

Taking the general equation of these surfaces when referred to their rectangular conjugates, viz.

$$Ax^2 + By^2 + Cz^2 = D \dots (1)$$

and substituting therein for x, y, z , the values in the formulas (A), at p. 198, we have, for the same surface, when related to oblique axes, the equation

$$\begin{aligned}
 A'x^2 + B'y^2 + C'z^2 + 2(A \cos X \cos X' + B \cos Y \cos Y' + C \cos \\
 Z \cos Z')xy \\
 + 2(A \cos X \cos X'' + B \cos Y \cos Y'' + C \cos \\
 Z \cos Z'')xz \\
 + 2(A \cos X' \cos X'' + B \cos Y' \cos Y'' + C \cos \\
 Z' \cos Z'')yz = D,
 \end{aligned}$$

in which equation A' , B' , C' , are put, for brevity, to represent certain functions of the cosines. Now the nine cosines which enter into this equation are subject to the three conditions (1), p. 198; and, in order that the three last terms of the equation may vanish, they must fulfil the additional conditions

$$A \cos X \cos X' + B \cos Y \cos Y' + C \cos Z \cos Z' = 0,$$

$$A \cos X \cos X'' + B \cos Y \cos Y'' + C \cos Z \cos Z'' = 0,$$

$$A \cos X' \cos X'' + B \cos Y' \cos Y'' + C \cos Z' \cos Z'' = 0.$$

Hence, if these six conditions have place, the cosines are such as to render the transformed equation of the form

$$A'x^2 + B'y^2 + C'z^2 = D \dots (2).$$

But to fix the values of these cosines, three more conditions are requisite, they being nine in number; which conditions, being arbitrary, may be infinitely varied; hence the position of the axes of reference may be infinitely varied, without altering the form of the equation (1), so that (260) the systems of conjugate diameters are infinite in number.

If any diameter be given, it will be easy to determine the diametral plane conjugate to it. For let the equations of any diameter be $x = mx$, $y = nz$; then putting (x', y', z') for its extremity, or

for the point where it meets the surface, we have

$$m = \frac{x'}{x}, n = \frac{y'}{y}.$$

Now from the equation of the tangent plane through (x', y', z') , given at page 183, it is obvious that the equation of the plane parallel to it, and passing through the centre, that is, of the diametral plane, is

$$\frac{x'x}{A^2} + \frac{y'y}{B^2} + \frac{z'z}{C^2} = 0,$$

or, dividing by x' and substituting the preceding values, the equation of the diametral plane is

$$\frac{mx}{A^2} + \frac{ny}{B^2} + \frac{z}{C^2} = 0.$$

Any diameter therefore being given, the plane, diametral to it, becomes fixed; but those diameters in this plane, which, with the given diameter, form a system of conjugates, are *not* fixed. For although the diameter which is given fixes the values of *three* of the cosines entering the equations of condition, viz. the values of $\cos X$, $\cos Y$, and $\cos Z$, there are still six others to be determined, yet there are but *five* equations to be satisfied;—the three above, and the last two of (1) at page 198. One of these six cosines is therefore still arbitrary, so that in the diametral plane we may draw a diameter at *any* inclination to one of the principal diameters; and after this is chosen, the direction of the third conjugate becomes fixed. This third conjugate, with that before chosen, necessarily forms a pair of conjugates in the section made by the diametral plane in which they are situated.

(293.) As to the *non-central* surfaces or those comprised in the general equation

$$px^2 + p'y^2 = pp'z,$$

it is easy to see from the preceding investigation that the transformed equation representing them will involve besides the products, two and two, of the variables, their simple powers; and hence, to reduce this transformed equation to the form above, we must satisfy in addition to the six equations of the preceding case two others, formed by equating the coefficients of x and y to zero. There will in this case therefore be *eight* equations of condition among *nine* undetermined quantities, still leaving one arbitrary, so that as before there is no limit to the number of systems of axes in reference to which the equation of the surface will preserve the same form. As already remarked, however, but *two* of the three planes whose intersections furnish any such system, are diametral (260).

The plane, diametral to any system of parallel chords, may be determined by the following method: let (x', y', z') denote the middle point of any one of the parallel chords; then, if the origin of the axes be transferred to this point, we shall have to substitute in the equation of the surface, instead of x, y, z , the values $x + x', y + y', z + z'$. But if the equations of the chord itself be $x = mz, y = nz$, then the coordinates of the points where it meets the surface will be found by substituting, instead of the foregoing, the following expressions for x, y, z in the original equation of the surface:—

$$mx + x', \quad nz + y', \quad z + z'.$$

The result of this substitution is

$$p(mz + x')^2 + p'(nz + y')^2 = pp'(z + z')$$

a quadratic which will furnish the *two* values of z belonging to the two extremities of the chord. As, however, these two values must be equal, and opposite in sign, seeing that the origin is at the middle of the chord, we infer that whichever chord may have been taken, the coefficient of z , in this quadratic, must be zero;

that is, the x', y', z' of the middle points of *every* system of parallel chords must fulfil the condition

$$2pmx' + 2p'ny' = pp',$$

so that the locus of these middle points must be the *plane* represented by this equation. It is evidently parallel to the axis of z , whatever be the values of m and n ; that is, whatever system of parallel chords be proposed. The mutual intersections of the diametral planes are therefore always diameters of the surface, being all parallel to the principal diameter.

The section of any diametral plane with the surface is a parabola, through any point of which, if a tangent plane be drawn, this will be the coordinate non-diametral plane at the point; an axis drawn in this plane from the point of contact, and parallel to the chords bisected by the diametral plane, will be a coordinate axis; the plane through this, and through the diameter of the parabola, will therefore be the second diametral plane; and thus, when one diametral plane is given, and also the point, on its parabolic section, where the other two coordinate planes are to meet it, these two are easily determined.

PROPOSITION IV.

(294.) In any central surface of the second order the longest of the principal diameters exceeds any other, and the shortest principal diameter is shorter than any other diameter.

Let A, B, C , be the principal semi-diameters of the surface, and of which A is either the longest or the shortest, then the equation of the surface is (260)

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1,$$

and the equation of a concentric sphere passing through the extremity of the semi-diameter, A , is

$$x^2 + y^2 + z^2 = A^2 \therefore x^2 = A^2 - y^2 - z^2.$$

If this sphere have any other points in common with the surface, the y, z , of those points will be given by substituting this value of x^2 in the above equation. This substitution gives

$$\left(\frac{1}{B^2} - \frac{1}{A^2}\right)y^2 + \left(\frac{1}{C^2} - \frac{1}{A^2}\right)z^2 = 0,$$

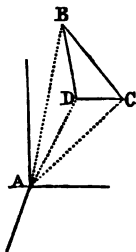
which equation is impossible, except in the single case $y = 0, z = 0$, since, by hypothesis, the two terms in the first member of this equation are either both negative or both positive, and therefore can never destroy each other.

PROPOSITION V.

(295.) To express the area of a triangle in space, by means of the coordinates of its vertices.

Since (263) the square of the area of any plane figure is equal to the sum of the squares of its projections on three rectangular planes, we shall be able to express the area of the triangle, provided we can express the areas of its projections in terms of the given coordinates.

Let BCD be the triangle in space, and draw lines from the origin, A , to its vertices B, C, D , of which the coordinates are x', y', z' ; x'', y'', z'' ; x''', y''', z''' . Now the projection of the triangle BCD on either of the coordinate planes is equal to the projection of ABC minus the projections of ABD, ACD , that is, for the projection of BCD , on the plane of xy , we have (22) Part I.



$$\frac{x'y'' - y'x''}{2} - \frac{x''y''' - y''x'''}{2} - \frac{x'y''' - y'x'''}{2}$$

or

$$\frac{1}{2}(x'y'' - y'x'' + y''x''' - x''y''' + y'x''' - x'y''').$$

In like manner, for the projections on the planes of xz, yz , we have the expressions

$$\frac{1}{2}(x'z'' - x'z'' + x''z''' - x''z''' + x'z''' - x'z'''),$$

and

$$\frac{1}{2}(y'z'' - y'z'' + y''z''' - y''z''' + y'z''' - y'z''').$$

Hence, calling this last expression $\frac{1}{2}A$, the preceding $\frac{1}{2}B$, and the first, $\frac{1}{2}C$, and putting a for the area of the proposed triangle, we have

$$a = \frac{\sqrt{A^2 + B^2 + C^2}}{2}$$

for the expression sought.

(296.) *Corollary.* If we determine the values A, B, C , in the equation

$$Ax + By + Cz = D$$

of the plane, passing through the three points $(x', y', z'), (x'', y'', z''),$

and (x''', y''', z''') , by means of the three equations of condition

$$Ax' + By' + Cz' = D,$$

$$Ax'' + By'' + Cz'' = D,$$

$$Ax''' + By''' + Cz''' = D,$$

we find for them precisely the expressions above, and for D we have

$$D = (x'y'' - y'x'')z''' + (y'z'' - z'y'')x''' + (z'x'' - x'z'')y''.$$

It appears, therefore, that the coefficients A , B , C , of the variables, in the equation of a plane passing through three points, denote the doubles of the projections of the triangle, whose vertices are these points, upon the planes of yz , zx , and xy .

PROPOSITION VI.

(297.) To express the equation of a plane by means of the perpendicular let fall upon it from the origin, and the inclinations of this perpendicular to the axes.

Representing the plane by the equation

$$Ax + By + Cz = D,$$

we have for the portions of the axes of x , y , z , intercepted between it and the origin, the respective values $\frac{D}{A}$, $\frac{D}{B}$, $\frac{D}{C}$. Now, if ρ be the perpendicular from the origin upon the plane, and α , β , γ , be the respective angles it makes with the axes of x , y , z , then

$$\rho = \frac{D}{A} \cos \alpha = \frac{D}{B} \cos \beta = \frac{D}{C} \cos \gamma,$$

Y

$$\therefore A = D \frac{\cos \alpha}{\rho}, B = D \frac{\cos \beta}{\rho}, C = D \frac{\cos \gamma}{\rho}.$$

Hence, substituting these values, in the equation of the plane, and multiplying by ρ , we have

$$x \cos \alpha + y \cos \beta + z \cos \gamma = \rho \dots (1)$$

for the equation sought.

When the plane passes through the origin, $\rho = 0$, and the equation is

$$x \cos \alpha + y \cos \beta + z \cos \gamma = 0 \dots (2).$$

PROPOSITION VII.

(298.) If the vertex of a pyramid be at the origin of three rectangular planes, and its base be projected upon them, then if any point be assumed in the plane of the base, the three pyramids whose bases are the projections, and vertices the assumed point, will together be equivalent to the original pyramid.

Let a represent the base of the proposed pyramid, and ρ a perpendicular upon it from the origin; then, if α, β, γ be the inclinations of this perpendicular to the axes of x, y, z , we shall have

$a \cos \alpha =$ the projection of a on the plane of yz ,

$a \cos \beta$ zx ,

$a \cos \gamma$ xy .

Now, by multiplying the equation of the plane (1), last proposition, by a , there results

$$xa \cos \alpha + ya \cos \beta + za \cos \gamma = a\rho \dots (1),$$

$$\therefore \frac{1}{3}xa \cos \alpha + \frac{1}{3}ya \cos \beta + \frac{1}{3}za \cos \gamma = \frac{1}{3}a\rho \dots (2).$$

The first member of this equation denotes three pyramids, whose bases are the projections above, and whose common vertex is any point (x, y, z) in the plane (1), last proposition, and whose perpendicular altitudes are respectively x, y , and z ; the second member represents the proposed pyramid; hence the truth of the theorem.

(299.) *Corollary 1.* Comparing equation (1), above, with $Ax + By + Cz = D$, since (prop. v. cor.) when a is a triangle, we have

$$A = 2a \cos \alpha, \quad B = 2a \cos \beta, \quad C = 2a \cos \gamma \therefore D = 2a\rho,$$

that is, the expression

$$(xy' - y'x'')x''' + (y'x'' - x'y''')x'' + (x'x'' - x''x''')y''' \dots (3)$$

represents six times the volume of the triangular pyramid, whose vertex is at the origin, and of which the corners of the base are the points (x', y', z') , (x'', y'', z'') , and (x''', y''', z''') , or, which is the same thing, the expression represents a parallelopiped, of which three contiguous edges meet at the origin and terminate in the points (x', y', z') , (x'', y'', z'') , (x''', y''', z''') .

(300.) *Cor. 2.* But those parts of the foregoing expression which are within the parentheses are obviously the projections of one of the faces of this parallelopiped, viz. that face whose contiguous sides terminate in the points (x', y', z') , (x'', y'', z'') , upon the three coordinate planes; and these projections are severally multiplied by the perpendiculars x''' , x'' , y''' , let fall upon them from the point (x''', y''', z''') . Hence the expression (3) represents the sum of the volumes of three parallelopipeds, having these projections for bases, and x''' , y''' , z''' , for altitudes.

(301.) *Cor. 3.* It follows, therefore, that if the vertex of a tri-

angular pyramid be at the origin of three rectangular planes, and either of its faces be projected on them, then the three pyramids constituted on these bases, and having a common vertex in that corner of the original pyramid's base, which is opposite to the projected face, shall together be equal to the original pyramid.

PROPOSITION VIII.

(302.) To determine the position of a plane, so that, if a given triangle be projected orthogonally upon it, the projection may be similar to a given triangle.

Let the plane of projection pass through a vertex of the given triangle, and let the perpendiculars, dropped from the other two vertices upon that plane, be x' , x'' ; let also the sides of the given triangle be A , B , C , and those of the triangle to which the projection is to be similar a , b , c ; then, on account of this similarity, the sides of the projected triangle will be

$$\sqrt{A^2 - x'^2}, \quad \frac{b}{a} \sqrt{A^2 - x'^2}, \quad \frac{c}{a} \sqrt{A^2 - x'^2}.$$

But the two latter projections are also

$$\sqrt{B^2 - x''^2}, \quad \sqrt{C^2 - (x' - x'')^2},$$

therefore

$$\frac{b^2}{a^2} (A^2 - x'^2) = B^2 - x''^2,$$

and

$$\frac{c^2}{a^2} (A^2 - x'^2) = C^2 - (x' - x'')^2.$$

Hence we have these two equations to find x' and x'' .

Substituting, in the second, the value of z'^2 , furnished by the first, we have, after transposing,

$$\frac{A^2c^2 - A^2b^2}{a^2} + B^2 - C^2 + \frac{a^2 + b^2 - c^2}{a^2} z'^2 = 2z' \sqrt{B^2 - \frac{A^2b^2}{a^2} + \frac{b^2}{a^2} z'^2}.$$

Squaring each side, and putting in the result single letters for the known coefficients, we have

$$p^2 + pqz'^2 + q^2z'^4 = rz'^2 + sz'^4,$$

$$\therefore z'^4 + \frac{pq-r}{q^2-s} z'^2 = -p^2.$$

This quadratic determines z' , and thence we get z'' , as also the three sides of the projected triangle, and thus the position of the required plane becomes known.

PROPOSITION IX.

(303.) If the equations

$$\left. \begin{aligned} a'^2 + b'^2 + c'^2 &= 1 \\ a''^2 + b''^2 + c''^2 &= 1 \\ a'''^2 + b'''^2 + c'''^2 &= 1 \end{aligned} \right\} \begin{aligned} a'a'' + b'b'' + c'c'' &= 0 \\ a'a''' + b'b''' + c'c''' &= 0 \\ a''a''' + b''b''' + c''c''' &= 0 \end{aligned} \dots (1)$$

exist, so also do the equations

$$\left. \begin{aligned} a'^2 + a''^2 + a'''^2 &= 1 \\ b'^2 + b''^2 + b'''^2 &= 1 \\ c'^2 + c''^2 + c'''^2 &= 1 \end{aligned} \right\} \begin{aligned} a'b' + a''b'' + a'''b''' &= 0 \\ a'c' + a''c'' + a'''c''' &= 0 \\ b'c' + b''c'' + b'''c''' &= 0 \end{aligned} \dots (2)$$

For, assume

$$\left. \begin{aligned} x &= a't + a''u + a'''v \\ y &= b't + b''u + b'''v \\ z &= c't + c''u + c'''v \end{aligned} \right\} \dots (3).$$

Then squaring each equation, and adding the results, we have in virtue of the proposed conditions

$$x^2 + y^2 + z^2 = t^2 + u^2 + v^2 \dots (4).$$

Let us now determine from (3) the values of t, u, v , in terms of x, y, z . In order to this, multiply the equations (3) respectively by a', b', c' , add the results, and we shall obtain t . Similarly, multiply by a'', b'', c'' , and we shall get u , &c. thus

$$\left. \begin{aligned} t &= a'x + b'y + c'z \\ u &= a''x + b''y + c''z \\ v &= a'''x + b'''y + c'''z \end{aligned} \right\} \dots (5).$$

Substitute these values in equation (4), and we shall have, by comparing the coefficients of the like terms, the equations announced.

PROPOSITION X.

(304.) If the conditions (1), in last proposition, exist, then also the following equations have place, viz.

$$\begin{aligned} (a'b' - b'a'')^2 &+ (a''b''' - b''a''')^2 + (a'''b' - b'''a')^2 = 1, \\ (b'c' - c'b'')^2 &+ (b''c''' - c''b''')^2 + (b'''c' - c'''b')^2 = 1, \\ (c'a' - a'c'')^2 &+ (c''a''' - a''c''')^2 + (c'''a' - a'''c')^2 = 1, \\ (a'b' - b'a'')c''' &+ (b'c' - c'b'')a''' + (c'a' - a'c'')b''' = 1. \end{aligned}$$

Put, for brevity, the first member of this last equation, $= l$; then if we determine the values of t, u, v , in equation (3), last proposition, not by the process there directed, but by the usual algebraical method (see *Algebra*, p. 78, where the operation is given at length,) we shall obtain these results, viz.

$$t = \frac{(b''c''' - c''b''')x + (b''c' - c''b')y + (b'c'' - c'b'')z}{l},$$

$$u = \frac{(c'a''' - a''c''')x + (c''a' - a''c')y + (c'a'' - a'c'')z}{l},$$

$$v = \frac{(a''b''' - b''a''')x + (a'''b' - b'''a')y + (a'b'' - b'a'')z}{l}.$$

Now these values must be identical with those marked (5), in the preceding investigation, provided the conditions there announced have place here.

Hence, comparing the coefficients of the like terms, we have

$$b''c''' - c''b''' = la', \quad b''c' - c''b' = la'', \quad b'c'' - c'b'' = la''',$$

$$c'a''' - a''c''' = lb', \quad c''a' - a''c' = lb'', \quad c'a'' - a'c'' = lb''',$$

$$a''b''' - b''a''' = lc', \quad a'''b' - b'''a' = lc'', \quad a'b'' - b'a'' = lc''.$$

Adding together the squares of the three equations in each horizontal row, in the last for example, we have, in virtue of the given conditions,

$$(a''b''' - b''a''')^2 + (a'''b' - b'''a')^2 + (a'b'' - b'a'')^2 = l^2.$$

It is easy to see that this equation may be put under the form

$$(a'^2 + a''^2 + a'''^2)(b'^2 + b''^2 + b'''^2) - (a'b' + a''b'' + a'''b''')^2 = l^2.$$

But, by the conditions (2), the first member of this equation is 1, therefore $l = 1$. Hence the truth of the first and fourth of the equations announced; and, by proceeding in like manner with the other two horizontal rows of equations above, we establish the truth of the two remaining equations announced.

PROPOSITION XI.

(305.) In a central surface of the second order the sum of the squares of any system of conjugate diameters is equivalent to the sum of the squares of the principal diameters.

Let A, B, C, represent the principal semi-diameters, and (x', y', z') , (x'', y'', z'') , (x''', y''', z''') , the extremities of any system of semi-conjugates A', B', C'. Then the equation of a tangent plane through the extremity of A', is (p. 183)

$$\frac{x'}{A^2}x + \frac{y'}{B^2}y + \frac{z'}{C^2}z = 1,$$

and this plane is parallel to both B' and C' (p. 186). Now the equations of B' are

$$x = \frac{x''}{x'''}z, \quad y = \frac{y''}{z'''}z,$$

and that this line may be parallel to the plane, we must have the condition (237)

$$\frac{x'x''}{A^2} + \frac{y'y''}{B^2} + \frac{z'z''}{C^2} = 0.$$

Hence we have

From the equation of the surface.

$$\frac{x'^2}{A^2} + \frac{y'^2}{B^2} + \frac{z'^2}{C^2} = 1$$

$$\frac{x''^2}{A^2} + \frac{y''^2}{B^2} + \frac{z''^2}{C^2} = 1$$

$$\frac{x'''^2}{A^2} + \frac{y'''^2}{B^2} + \frac{z'''^2}{C^2} = 1$$

From the equation of the tangent planes.

$$\frac{x'x''}{A^2} + \frac{y'y''}{B^2} + \frac{z'z''}{C^2} = 0$$

$$\frac{x''x'}{A^2} + \frac{y''y'}{B^2} + \frac{z''z'}{C^2} = 0$$

$$\frac{x'x'''}{A^2} + \frac{y'y'''}{B^2} + \frac{z'z'''}{C^2} = 0.$$

Consequently, (prop. ix.),

$$x'^2 + x''^2 + x'''^2 = A^2$$

$$y'^2 + y''^2 + y'''^2 = B^2$$

$$z'^2 + z''^2 + z'''^2 = C^2$$

Adding these equations, $A'^2 + B'^2 + C'^2 = A^2 + B^2 + C^2$.

PROPOSITION XII.

(306.) In a central surface of the second order the sum of the squares of the faces of the parallelopiped whose edges are any system of semi-conjugate diameters, is equal to the sum of the squares of the faces of the rectangular parallelopiped whose edges are the semi-principal diameters; also the volume of the former is equal to the volume of the latter.

Since the conditions furnished by the equations of the surface and by the equations of the tangent planes at the extremities of the

conjugate diameters, as exhibited in last proposition, agree with the conditions in prop. ix., $a' b', c'$, being here replaced by $\frac{x'}{A}, \frac{y'}{B}, \frac{z'}{C}$,

&c., we may derive from these conditions the equations announced in prop. x., which, in the present case, are

$$(x'y'' - y'x'')^2 + (x''y''' - y''x''')^2 + (x'''y' - y'''x')^2 = A^2 B^2$$

$$(y'x'' - x'y'')^2 + (y''x''' - x''y''')^2 + (y'''x' - x'''y')^2 = B^2 C^2$$

$$(x'x'' - x'x'')^2 + (x''x''' - x'x''')^2 + (x'''x' - x'x')^2 = A^2 C^2$$

$$(x'y'' - y'x'')x''' + (y'x'' - x'y'')x''' + (x'x'' - x'x'')y''' = ABC.$$

Now the first vertical row of terms in the first three equations exhibits the sum of the squares of the projections of the parallelogram whose contiguous sides are A', B' ; the second vertical row is the sum of the squares of the projections of the parallelogram whose sides are B', C' , and the third row is the sum of the squares when the sides are A', C' . Hence, by adding the three equations together, it follows (263) that the sum of the squares of the sides of the parallelopiped whose edges are A', B', C' , is equal to the sum of the squares of the sides of the parallelopiped whose edges are A, B, C . Again, the first member of the fourth equation expresses the volume of the parallelopiped whose edges are A', B', C' , (prop. vii., cor. 1); hence this parallelopiped is equal to that whose edges are A, B, C .

Cor. From this theorem we may immediately infer that there can be but one system of rectangular conjugates except the surface be of revolution. For, if A', B', C' , could be mutually at right angles, as well as A, B, C , then from these theorems, and the composition of equations, the equations

$$(x + A^2)(x + B^2)(x + C^2) = 0$$

and

$$(x + A^2)(x + B'^2)(x + C'^2) = 0$$

would be identical, and their roots the same; therefore A, B, C , are respectively equal to A', B', C' , which (prop. iv.) is impossible, when the surface is not of revolution.

PROPOSITION XIII.

(307.) In a central surface of the second order the squares of the reciprocals of any system of rectangular diameters are together equal to the squares of the reciprocals of the principal diameters.

Let A, B, C , be the principal semi-diameters of the surface, and A', B', C' , any other semi-diameters mutually at right angles. Then the equation of the surface is

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1.$$

To transform this equation, so as to refer the surface to another system of axes, we must substitute for x, y, z , the values (A) , (p. 198), which gives

$$\left(\frac{\cos^2 X}{A^2} + \frac{\cos^2 Y}{B^2} + \frac{\cos^2 Z}{C^2}\right) x^2 + \left(\frac{\cos^2 X'}{A^2} + \frac{\cos^2 Y'}{B^2} + \frac{\cos^2 Z'}{C^2}\right) y^2 + \left(\frac{\cos^2 X''}{A^2} + \frac{\cos^2 Y''}{B^2} + \frac{\cos^2 Z''}{C^2}\right) z^2 + Pxy + Qxz + Ryz = 1,$$

where P, Q, R , are put to represent certain functions of the inclinations of the new axes to the old.

Now, to determine the lengths of the semi-diameters, which coincide with these new axes, put successively, in this equation,

$$z = 0, y = 0; \quad z = 0, x = 0; \quad x = 0, y = 0;$$

and there results

$$\frac{1}{x^2} = \frac{1}{A^2} = \frac{\cos^2 X'}{A^2} + \frac{\cos^2 Y}{B^2} + \frac{\cos^2 Z}{C^2}$$

$$\frac{1}{y^2} = \frac{1}{B^2} = \frac{\cos^2 X'}{A^2} + \frac{\cos^2 Y'}{B^2} + \frac{\cos^2 Z'}{C^2}$$

$$\frac{1}{z^2} = \frac{1}{C^2} = \frac{\cos^2 X''}{A^2} + \frac{\cos^2 Y''}{B^2} + \frac{\cos^2 Z''}{C^2}$$

$$\text{Adding, } \frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2} = \frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2}$$

because, by hypothesis, the two systems of axes are rectangular, and (221) the sum of the squares of the cosines of the inclinations of any straight line to three rectangular axes is always unity.

PROPOSITION XIV.

(308.) Three rectangular planes constantly touch a central surface of the second order; required the locus of their point of intersection.

Let the equations of the surface and of a tangent plane at the point (x', y', z') be

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1$$

$$\frac{x'x}{A^2} + \frac{y'y}{B^2} + \frac{z'z}{C^2} = 1;$$

then, putting successively

$$y = 0, z = 0; x = 0, z = 0; x = 0, y = 0;$$

we have for the parts of the axes included between this plane and the origin, the values

$$\frac{A^2}{x'}, \frac{B^2}{y'}, \frac{C^2}{z'}.$$

Hence, if ρ be the perpendicular from the origin upon the plane, and if α, β, γ , denote the angles it makes with these three lines, we have

$$\rho = \frac{\cos \alpha A^2}{x'} = \frac{\cos \beta B^2}{y'} = \frac{\cos \gamma C^2}{z'}$$

$$\therefore \cos \alpha = \frac{\rho x'}{A^2}, \cos \beta = \frac{\rho y'}{B^2}, \cos \gamma = \frac{\rho z'}{C^2}$$

$$\therefore A^2 \cos^2 \alpha + B^2 \cos^2 \beta + C^2 \cos^2 \gamma = \rho^2 \left(\frac{x'^2}{A^2} + \frac{y'^2}{B^2} + \frac{z'^2}{C^2} \right) = \rho^2.$$

In a similar manner, if ρ', ρ'' , be perpendiculars from the origin upon two tangent planes at the points (x', y', z') and (x'', y'', z'') , and if the first make angles α', β', γ' , with the axes, and second make the angles $\alpha'', \beta'', \gamma''$, we have

$$A^2 \cos^2 \alpha' + B^2 \cos^2 \beta' + C^2 \cos^2 \gamma' = \rho'^2$$

$$A^2 \cos^2 \alpha'' + B^2 \cos^2 \beta'' + C^2 \cos^2 \gamma'' = \rho''^2.$$

Now if the three tangent planes be mutually rectangular, ρ, ρ', ρ'' , will be mutually rectangular; so that the sum of the squares of the cosines which they make with the axis of x , with the axis of y , and with the axis of z , is in either case equal to unity (221). Hence, by addition,

$$A^2 + B^2 + C^2 = \rho^2 + \rho'^2 + \rho''^2.$$

If R represent the distance of the intersection (x, y, z) of the tangent planes from the origin, then $R^2 = \rho^2 + \rho'^2 + \rho''^2$; but $R^2 = x^2 + y^2 + z^2$; hence

$$x^2 + y^2 + z^2 = A^2 + B^2 + C^2$$

is the equation of the locus of (x, y, z) which is a sphere concentric with the proposed surface, and of which the radius is

$$R = \sqrt{A^2 + B^2 + C^2}.$$

PROPOSITION XV.

(309.) Chords are drawn to a surface of the second order so as all to pass through a fixed point; what is the locus of their middle points?

Assume the fixed point for the origin, let the axis of x pass through the centre of the surface, if it have one, or be parallel to a diameter if it have not, and let the other two be parallel to the two axes conjugate to this, then the equation of the surface will be

$$Ax^2 + By^2 + Cz^2 + Fx = G,$$

and the equations of any chord through the origin are $x = mz$, $y = nz$; substituting these values in the equation of the surface, we have at the points common to both

$$(A + Bn^2 + Cm^2)z^2 + Fmz = G,$$

and half the sum of the two values of z given by this equation must be the z of the middle of the chord, that is, by the theory of equations, this z is

$$z = -\frac{\frac{1}{2}Fm}{A + Bn^2 + Cm^2};$$

or, substituting for m and n the values $\frac{x}{z}, \frac{y}{z}$, as given by the equations of the chord, we have, after reduction,

$$Ax^2 + By^2 + Cz^2 + \frac{1}{2}Fx = 0,$$

for the equation of the locus, which is, therefore, a surface of the second order, similar to the proposed.* If, in this equation, we make any two of the variables 0, we have 0 for one value of the third; thus showing that the surface, if completed, would pass through the origin. The coordinates of the centre of the proposed surface are (p. 225)

$$x' = -\frac{F}{2C}, y' = 0, z' = 0,$$

and these same coordinates satisfy the equation of the locus; hence the locus passes through the centre of the proposed surface, if it have a centre.

If we subtract the equation of the locus from that of the proposed surface, there results

$$\frac{1}{2}Fx = G \therefore x = \frac{2G}{F};$$

this therefore is the value of the abscissa x belonging to every point common to both surfaces; consequently the two surfaces intersect

in a plane parallel to the plane of yz , and at the distance $\frac{2G}{F}$ from the fixed point. If this point be upon the proposed surface, then $G = 0$; hence, in that case, the two surfaces merely touch at that point.

* See NOTE B, at the end.

PROPOSITION XVI.

(310.) Planes passing through a fixed point cut a surface of the second order; what is the locus of the centres of all the sections?

It is not difficult to perceive that the locus will be in this case the same as in last proposition, but we shall give an independent investigation.

Assuming the same axes as before, we have, for the equations of the surface, and of any plane through the origin,

$$Ax^2 + By^2 + Cz^2 + Fx = G \dots (1)$$

$$z = mx + ny \dots (2).$$

Substituting this value of z , in the first equation, we have for the x, y of the intersection the equation

$$(Am^2 + C)x^2 + 2Amnxy + (An^2 + B)y^2 + Fx = G \dots (3).$$

Let the x, y , of the centre of this section be x', y' , then, if the origin be removed to this centre, x, y , must be changed into $x + x', y + y'$, which changes (3) into

$$(Am^2 + C)(x + x')^2 + 2Amn(x + x')(y + y') + (An^2 + B)(y + y')^2 + F(x + x') = G.$$

Now the origin being by hypothesis at the centre, the coefficients of x and y must vanish from this equation. These coefficients, without developing all the terms, are readily seen to be

$$2Am(mx' + ny') + 2Cx' + F = 0$$

$$2Ax(mx' + ny') + 2By = 0.$$

that is, from equation (2),

$$\left. \begin{aligned} 2Amx' + 2Cx' + F &= 0 \\ 2Anx' + 2By' &= 0 \end{aligned} \right\} \therefore \begin{cases} m = -\frac{Cx' + \frac{1}{2}F}{Ax'} \\ n = -\frac{By'}{Ax'}. \end{cases}$$

These values of m and n , substituted in the value of x' , (2), give

$$Ax'^2 + By'^2 + Cx'^2 + \frac{1}{2}Fx' = 0$$

for the equation of the locus, which is the same as that deduced in last prop., and to which, therefore, the same remarks apply.*

PROPOSITION XVII.

(311.) Three straight lines mutually at right angles meet in a point and constantly touch a surface of the second order; what is the locus of the point?

Let the equation of the surface be

$$Ax^2 + By^2 + Cz^2 = D,$$

and let x', y', z' , be the coordinates of a point in the locus, then, if the origin be removed to this point, and the touching lines be taken

* A very simple *geometrical* solution of this problem is given in the Gentleman's Diary for 1830, by *Professor Davies*, of the Royal Military Academy. We take this opportunity of recommending to the student's attention the very instructive researches on *Geometry of three Dimensions*, which this gentleman has published in xx., xxi., and subsequent numbers of *Leybourn's Repository*.

for axes, the equation of the surface will (269) be transformed into

$$\begin{array}{ccc}
 A \cos^2 X & \left| \begin{array}{c} x^2 + A \cos^2 X' \\ B \cos^2 Y \\ C \cos^2 Z \end{array} \right| & \left| \begin{array}{c} y^2 + A \cos^2 X'' \\ B \cos^2 Y' \\ C \cos^2 Z' \end{array} \right| & \left| \begin{array}{c} z^2 \\ B \cos^2 Y'' \\ C \cos^2 Z'' \end{array} \right| \\
 + A \cos Xx' & \left| \begin{array}{c} 2x + A \cos X'x' \\ B \cos Yy' \\ C \cos Zz' \end{array} \right| & \left| \begin{array}{c} 2y + A \cos X''x' \\ B \cos Y'y' \\ C \cos Z'x' \end{array} \right| & \left| \begin{array}{c} 2z \\ B \cos Y''y' \\ C \cos Z''x' \end{array} \right| \\
 + Pxy + Qxz + Ryz = D - Ax'^2 - By'^2 - Cz'^2
 \end{array}$$

where P, Q, R, are put for brevity to represent certain functions of the cosines. Now, in order to determine the parts of the axes intercepted between the origin and the surface, we must put successively in this equation

$$y = 0, z = 0; \quad x = 0, z = 0; \quad x = 0, y = 0;$$

and there results

$$(A \cos^2 X + B \cos^2 Y + C \cos^2 Z) x^2 + 2 (A \cos Xx' + B \cos Yy' + C \cos Zz') x + D' = 0$$

$$(A \cos^2 X' + B \cos^2 Y' + C \cos^2 Z') y^2 + 2 (A \cos X'x' + B \cos Y'y' + C \cos Z'x') y + D' = 0$$

$$(A \cos^2 X'' + B \cos^2 Y'' + C \cos^2 Z'') z^2 + 2 (A \cos X''x' + B \cos Y''y' + C \cos Z''x') z + D' = 0,$$

where D' is put for $Ax'^2 + By'^2 + Cz'^2 - D$. These equations furnish two values for each of the quantities x, y, z , corresponding to the *two* points in which each axis *cuts* the surface; but, if we introduce the conditions that each axis merely *touches* the surface, the two points coincide; and, therefore, in this case, the two roots

of each equation become equal. Hence, by the theory of equations,

$$D'(A \cos^2 X + B \cos^2 Y + C \cos^2 Z) = (A \cos X x' + B \cos Y y' + C \cos Z z')^2$$

$$D'(A \cos^2 X' + B \cos^2 Y' + C \cos^2 Z') = (A \cos X' x' + B \cos Y' y' + C \cos Z' z')^2$$

$$D'(A \cos^2 X'' + B \cos^2 Y'' + C \cos^2 Z'') = (A \cos X'' x' + B \cos Y'' y' + C \cos Z'' z')^2.$$

Adding these equations together, we have, in virtue of the conditions, (1) and (3), pp. 198, 199,

$$D'(A + B + C) = A^2 x'^2 + B^2 y'^2 + C^2 z'^2,$$

that is,

$$(Ax'^2 + By'^2 + Cz'^2 - D)(A + B + C) = A^3 x'^2 + B^3 y'^2 + C^3 z'^2$$

whence

$$A(B + C)x'^2 + B(A + C)y'^2 + C(A + B)z'^2 = D(A + B + C)$$

the equation of the locus, which is a surface of the second order concentric with the proposed.

PROPOSITION XVIII.

(312.) If an ellipsoid be cut by a plane through its centre, the difference of the squares of the reciprocals of the semi-axes of the section, will be proportional to the product of the sines of the angles which the plane of the section makes with the planes of the two circular sections of the ellipsoid.

Call the semi-axes of the ellipsoid A, B, C ; A, B , being those

of the greatest section, and which we shall suppose to coincide with the plane of the paper; then the principal section containing the greatest and least semi-axes A, C , will be vertical. Let the principal semi-diameters of any oblique section through the centre be a, b . This oblique section will make traces a', a_1 , on each of the equal circular sections, which will be the radii of those sections; and as these equal traces are also semi-diameters of the oblique section, the semi-diameter b , of that section, must bisect the angle $[a', a_1]$.

We may thus conceive, as situated above the plane of the paper, and issuing from the centre of the ellipsoid, these four lines: viz. the minor semi-diameter, C , and the three lines a', b, a_1 ; these latter being all in the plane of the oblique section, and the middle one bisecting the angle between the other two.

Disregarding, for the present, one of the extreme lines a_1 , let us attend to the other two, a', b ; which, in conjunction with the horizontal semi-axis B , form the edges of a solid angle at the centre; and which will pierce a sphere, described about that centre, in the three vertices of a spherical triangle, whose sides measure the angles $[a', b]$, $[a', B]$, $[b, B]$, between the edges; and whose angles measure the inclinations of the planes. Call the angle between the planes of $[a', b]$ and $[a', B]$, that is, the inclination of the oblique section to the circular section in which a' is, I . This is evidently the spherical angle at the point where a' pierces the sphere. The inclination of the planes of $[a', B]$ and $[b, B]$, since they are evidently both perpendicular to the vertical principal plane, is measured by the angle between their traces upon this vertical plane; the trace of the first plane—that of the circular section—is of course a' : call the trace of the other C' . Then the angle $[a', C']$ measures the spherical angle at the point where B pierces the sphere.

It hence appears that the three sides of our spherical triangle, and those two angles of it that we have fixed upon, are measured and posited with respect to each other, as follow:

Sides	$[a', b]$	$[a', B]$	$[b, B]$
Angles	I	$[a', C']$	

so that I is opposite to the side $[b, B]$; and $[a', C']$ opposite to the side $[a', b]$. And, as the sines of the sides are as the sines of the opposite angles, we have

$$\sin I = \frac{\sin [b, B]}{\sin [a', b]} \sin [a', C'].$$

If the trace a_1 of the oblique section upon the other circular section had been taken for an edge of the solid angle instead of a' , the corresponding spherical triangle would have given, in like manner, for the inclination I_1 ,

$$\sin I_1 = \frac{\sin [b, B]}{\sin [a_1, b]} \sin [a_1, C']$$

and, as the angles $[a', b]$, $[a_1, b]$ are equal, we have

$$\sin I \sin I_1 = \frac{\sin^2 [b, B]}{\sin^2 [a', b]} \sin [a', C'] \sin [a_1, C'] \dots (1)$$

for the product of the sines of the angles which the oblique section makes with the circular sections.

Again, by art. (56), Part I., the length of any semi-diameter A' of an ellipse is

$$\begin{aligned} A'^2 &= B^2 + e^2 x^2 \\ &= B^2 + \frac{A^2 - B^2}{A^2} A'^2 \cos^2 [A, A'] \\ &= B^2 + \left(1 - \frac{B^2}{A^2}\right) A'^2 \sin^2 [B, A']. \end{aligned}$$

Therefore, dividing by A'^2B^2 and transposing, we have

$$\frac{1}{B^2} - \frac{1}{A'^2} = \left(\frac{1}{B^2} - \frac{1}{A^2} \right) \sin^2 [B, A'] \dots (2)$$

or, putting for $\sin^2 [B, A']$ its equal $1 - \sin^2 [A, A']$, and transposing,

$$\frac{1}{B'^2} - \frac{1}{A'^2} = \left(\frac{1}{B^2} - \frac{1}{A^2} \right) \sin^2 [A, A'] \dots (3).$$

Hence, in the ellipse, whose principal semi-diameters are B, C' ; and in which the oblique semi-diameter b is situated, we have (3)

$$\frac{1}{b^2} - \frac{1}{B^2} = \left(\frac{1}{C'^2} - \frac{1}{B^2} \right) \sin^2 [b, B].$$

In the ellipse, whose principal semi-diameters are a, b ; and in which a' is situated, we have (2)

$$\frac{1}{b^2} - \frac{1}{a'^2} = \left(\frac{1}{b^2} - \frac{1}{a^2} \right) \sin^2 [a', b].$$

Consequently, since $a' = B$, art. (283),

$$\frac{1}{b^2} - \frac{1}{a^2} = \left(\frac{1}{C'^2} - \frac{1}{B^2} \right) \frac{\sin^2 [b, B]}{\sin^2 [a', b]} \dots (4).$$

But from the vertical principal ellipse, or that whose semi-axes are A, C , and in which are the three oblique semi-diameters a', C', a_1 , — the first and third of which are equally inclined to the diameter $2A$, — we have (3)

$$\frac{1}{C^2} - \frac{1}{A^2} = \left(\frac{1}{C^2} - \frac{1}{A^2} \right) \sin^2 [A, C]$$

$$\frac{1}{a'^2} - \frac{1}{A^2} = \left(\frac{1}{C^2} - \frac{1}{A^2} \right) \sin^2 [A, a'].$$

$$\text{Subtracting} \quad \frac{1}{C^2} - \frac{1}{B^2} = \left(\frac{1}{C^2} - \frac{1}{A^2} \right) (\sin^2 [A, C] - \sin^2 [A, a'])$$

remembering that $a' = B$. But (*Trigonometry*, p. 47) the difference of the squares of the sines of two arcs is equal to the product of the sines of the sum and difference of those arcs. The difference of the two arcs $[A, C]$, $[A, a']$ is evidently $[a', C]$; and, on account of the equal inclinations of a' and a_1 to the major diameter $2A$, the supplement of their sum is $[a_1, C]$. Therefore, the sine of this being the same as the sine of the sum itself, we have

$$\frac{1}{C^2} - \frac{1}{B^2} = \left(\frac{1}{C^2} - \frac{1}{A^2} \right) \sin^2 [a', C] \sin [a_1, C].$$

Hence, by substitution in (4),

$$\begin{aligned} \frac{1}{b^2} - \frac{1}{a^2} &= \left(\frac{1}{C^2} - \frac{1}{A^2} \right) \frac{\sin^2 [b, B]}{\sin^2 [a', b]} \sin [a', C] \sin [a_1, C] \\ &= \left(\frac{1}{C^2} - \frac{1}{A^2} \right) \sin I \sin I_1, \text{ by equation (1).} \end{aligned}$$

This property is of importance in the Undulatory Theory of Light. The foregoing investigation of it is in substance the same as that given by Professor M'Cullagh, in his very interesting paper on "Double Refraction in a Crystalline Medium," in the *Transactions of the Royal Irish Academy*, vol. xvi.

PROPOSITION XIX.

(313.) If N represent the normal at any point of the earth's surface, supposed to be an oblate spheroid, and if λ denote the latitude or angle under the normal, and equatorial diameter $2A$, prove that

$$N = \frac{A(1 - e^2)}{\sqrt{1 - e^2 \sin^2 \lambda}}.$$

PROPOSITION XX.

If N be produced to meet the polar diameter, show that the whole length, R , is

$$R = \frac{A}{\sqrt{1 - e^2 \sin^2 \lambda}}.$$

PROPOSITION XXI.

If P represent the perpendicular from the centre of an ellipsoid on a tangent plane, prove that

$$\frac{1}{P^2} = \frac{x^2}{A^4} + \frac{y^2}{B^4} + \frac{z^2}{C^4}.$$

PROPOSITION XXII.

If the semi-axes A, B, C and A', B', C' of two concentric ellipsoids coincide in direction, and be reciprocally proportional, so that $AA' = BB' = CC' = k^2$; and if any semi-diameter of one

be cut by a plane which touches the other, the rectangle under the whole semi-diameter, and that part of it between the centre and tangent plane, will always be equal to k^2 .*

PROPOSITION XXIII.

If through any system of conjugate diameters, in a principal section of an ellipsoid, planes perpendicular to that section pass, prove that the squares of the sections made by these planes will always amount to the same sum.

PROPOSITION XXIV.

If a conical surface envelope a surface of the second order, prove that the curve of contact is of the second order.

PROPOSITION XXV.

Parallel planes cut a surface of the second order; required the locus of the foci of the sections.

PROPOSITION XXVI.

Given the position of two lights of known intensities (m, n), to determine the surface of which every point shall be equally illuminated by both lights, the law of intensity varying inversely as the square of the distance.

* This property of *reciprocal ellipsoids* is from the Transactions of the Royal Irish Academy before referred to; and is employed by Mr. M'Cullagh, to determine geometrically the form of *Fresnel's Wave Surface* in the Undulatory Theory of Light. The proof of the property is very easily made out from the expressions at page 253.

(314.) Before closing the present volume, we shall very briefly advert to a class of curves denominated *curves of double curvature*, a name by which all lines are designated which cannot be traced upon a plane but only upon a curve surface. The simplest analytical representation of such a line is analogous to that already employed for the straight line in space, viz. by the combination of the two equations denoting the two surfaces which project the proposed line on two coordinate planes. In the case of the straight line, these projecting surfaces are planes; in lines of double curvature they are obviously cylindrical surfaces. The equations to a curve of double curvature are, therefore, those two combined which, taken separately, represent the two projecting cylinders.

A line of double curvature generally presents itself, in mathematical enquiries, as the intersection of two curve surfaces, and for these surfaces we are, for simplicity, to substitute the two intersecting cylinders of which we have just spoken. This is effected thus: let one of the variables, as z , be eliminated from the equations of the two intersecting surfaces, and there will result a function of x and y as $F(x, y) = 0$; and this represents the cylinder projecting the intersection on the plane of xy . In like manner, eliminate another variable as y , and we have $f(x, z) = 0$ for the representative of the cylinder, projecting the same curve on the plane of xz ; so that the equations sought are

$$F(x, y) = 0, \quad f(x, z) = 0.$$

Suppose, for example, it were required to express the equations of the curve of double curvature formed by the intersection of a sphere, which is pierced through its centre by a right cylinder, supposing the diameter of the base to be equal to the radius of the sphere. Taking the centre of the sphere for the origin, the line coinciding with the cylindrical surface for the axis of z , and the perpendicular to this touching the same surface, for the axis of y , we have, since the given cylinder projects the curve of intersection on the plane of xy into a circle whose diameter is r , this equation to the projection, viz.

$$y^2 = rx - x^2 \dots (1),$$

which also expresses the relation between the x, y , of every point in the cylinder, and consequently between the x, y , of every point on the sphere belonging to the intersection. The equation of this sphere is

$$x^2 + y^2 + z^2 = r^2;$$

hence the relation between the x, z , of the intersection is (1),

$$z^2 = r^2 - rx \dots (2).$$

which, therefore, is the equation of the projection on the plane of xz , consequently the proposed curve of double curvature is expressed analytically by the equations (1) and (2) combined, viz.

$$\left. \begin{aligned} y^2 &= rx - x^2 \\ z^2 &= r^2 - rx \end{aligned} \right\}.$$

If we had to determine the tangent line at any point in a curve of double curvature, we should first determine the two tangents through the projections of that point, then perpendicular planes through these tangents would obviously be in contact with the two projecting cylinders throughout the lengths of these cylinders, and would each pass through the proposed point in their intersection, and through no other; hence the intersection of these planes would be the linear tangent sought, and would, therefore, be analytically represented by the two equations, which, separately, represent the tangents to the projections.

We cannot enter into further particulars respecting these curves here, since their full discussion requires the aid of the transcendental analysis; but the preceding remarks may serve to convey a notion of this class of lines and of the manner of representing them by equations.

(315.) It may be proper, however, to remark in conclusion, that although, as stated above, a curve of double curvature is the usual result of the intersection of two curve surfaces; yet the curve of intersection is sometimes plane; and it is a remarkable fact that

when one surface of the second order is penetrated by another, forming two distinct intersections, if one of these is a plane curve, the other must be a plane curve also.

For, suppose both surfaces to be referred to the same coordinate planes, of which one, as the plane of xy , coincides with the intersection admitted to be plane. The two surfaces may be represented by the equations

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fxy + Gz + Hy + Kx + L = 0$$

$$A'x^2 + B'y^2 + C'z^2 + D'xy + E'xz + F'xy + G'z + H'y + K'x + L' = 0$$

which are *simultaneous* at the intersections. One of these intersections being in the plane of xy , it is obvious, that if we put $z = 0$ in each of the above equations, the resulting equations in x, y must be identical, because each will represent the same curve.

These equations are

$$By^2 + Cz^2 + Fxy + Hy + Kx + L = 0$$

$$B'y^2 + C'z^2 + F'xy + H'y + K'x + L' = 0;$$

and, in consequence of their both representing the same thing, the coefficients of one can differ from those corresponding to them in the other, only by a common numerical factor, m . Hence

$$B = mB', C = mC', F = mF'$$

$$H = mH', K = mK', L = mL';$$

such, then, are the conditions by which the coefficients in the two proposed equations must be connected, in order that the plane of xy may be that of an intersection.

This being settled, let us multiply the second of the proposed equations by m , and subtract it from the first; the result, which can belong only to those points on each surface which are common to both, will, by the foregoing relations, be

$$(A - mA')x^2 + (D - mD')xy + (E - mE)xz + (G - mG')z = 0;$$

and this is equivalent to the two distinct equations

$$z = 0 \dots \dots (1)$$

$$(A - mA')z + (D - mD')y + (E - mE)x + G - mG' = 0 \dots (2),$$

the first showing that one series of points common to both surfaces is situated in the plane of xy , and the second showing that another series is situated also in a plane, seeing that the coordinates of each point satisfy the equation (2) of a plane.

For further researches into the theory of curves and surfaces in general, the student may consult the treatise on the *Differential Calculus*, section iii.; and for a systematic development of the principles of a new species of Analytical Geometry—the geometry of *Spherical Coordinates*—he is referred to the very original and interesting paper of Professor Davies, “On the Equations of Loci traced upon the Surface of the Sphere,” in vol. xii. of the *Transactions of the Royal Society of Edinburgh*.

NOTES.

NOTE A. Page 6.

On the Varieties of the Ellipse and Hyperbola.

THE varieties of the three curves enumerated in the text comprehend all the modifications usually distinguished by that name. But for the same reason that a point, into which a circle ultimately merges by the continual diminution and ultimate evanescence of its radius, is called a variety of the circle, ought the finite straight line, into which the ellipse passes when one of its axes by continually diminishing ultimately vanishes, be regarded as a variety of the ellipse. In the text the equation, which, in its ordinary

form, represents an ellipse, is reduced to the representative of a point by undergoing these mutations:—first, the coefficients of x^2 and of y^2 become equal; and the ellipse in consequence changes into a circle; next these equal coefficients simultaneously vanish, and the circle merges into a point. But instead of this succession of hypotheses, let us assume the single condition, that *one* only of the aforesaid coefficients, by continually diminishing, ultimately vanishes; while the other remains unaffected by these variations. The curve must then degenerate, not into a point, but into a straight line, limited in both directions; it must in fact collapse into the fixed axis. The proper analytical representation of this extreme case—supposing A to be the invariable axis—is

$$\left. \begin{aligned} A^2y^2 + B^2x^2 &= A^2B^2 \\ B &= 0 \end{aligned} \right\};$$

for we thus preserve the controlling influence, to which every particular case must be subject,—even the extreme case here selected. It is of great importance to the student of analysis, that he have a clear apprehension of these ultimate cases, or ultimate values, and of the restrictions under which every such value must come, in common with each of the entire series of values that may have preceded it; bearing in mind, that at the instant any case arrives at the utmost boundary of the restrictions involved in the premises, it is equally the *last* of the admissible cases, and the first of those in the unlimited region beyond; and that consequently at such critical stages caution becomes especially requisite. The term *limit*, however, as applied to such cases, is often ill explained by the teacher, and ill understood by the student. It is a term that there would be no use in employing did we not mean by it, that the extreme value, to which it refers, is to be governed by the very same conditions that affect each individual of the whole series of results, of which series that selected is either the origin or the termination.

In the instance before us an inseparable connexion between x and y is fixed by the original equation; a connexion which must not be violated in any individual case involved in that equation: y *must* be impossible for $x > A$; and x *must* be impossible for $y > B$. If these restrictions be overstepped in any particular instance, then that case cannot be one of the

series of cases comprehended in the proposed equation ; the expression for y , independently of particular values for A and B , is

$$y = \frac{B}{A} \sqrt{A^2 - x^2}.$$

This, when $B = 0$ becomes, unquestionably, $y = 0$, whatever be the value of x . Let x exceed A , that is, let $x^2 = A^2 + m$; the resulting expression for y is $y = \pm 0 \sqrt{-m}$; the geometrical interpretation of which evidently is, that the point, furnished by the above equation for an abscissa $x > A$, is upon the axis of x , but in the imaginary region—beyond the bounds of the locus. If B had been fixed, and A had merged into zero, the line represented would obviously have been the minor diameter $2B$.

The extreme case of the hyperbola, corresponding to the case $B = 0$ of the ellipse, will be two unlimited straight lines, proceeding in opposite directions, and separated by the interval $2A$, which interval, diminishing with A , will disappear when A becomes zero; and the two lines, thus meeting, will become infinite in both directions. The variety in the text—two intersecting straight lines—arises from the simultaneous evanescence of *both* axes; their *ratio*, however, remaining fixed.—See page 4.

NOTE B. Page 255.

On Similar Curves and Surfaces of the Second Order.

WHEN two indeterminate equations of the second order have the coefficients of the like terms of two dimensions in a constant ratio; that is, when the two equations are

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0$$

$$mAy^2 + mBxy + mCx^2 + D'y + E'x + F' = 0,$$

the curves represented by them are *similar*. For, by referring to page 9, we see that this constant ratio m , between the corresponding coefficients